EULER CLASSES & NEGATIVE POWERS OF THE CANONICAL CLASS

§1) Moduli of Curves

nodal singularities

$$\mathcal{M}_{g,n} := \left\{ (C, p_1, \dots, p_n) \;\middle|\; \begin{array}{l} C \text{ Riemann surface} \\ \text{genus} = g \\ p_1, \dots, p_n \in C \text{ mrk pnts} \end{array} \right\} \Big/ \sim \;\subseteq\; \overline{\mathcal{M}}_{g,n}$$

smooth $\mathbb{C}$-orbifold

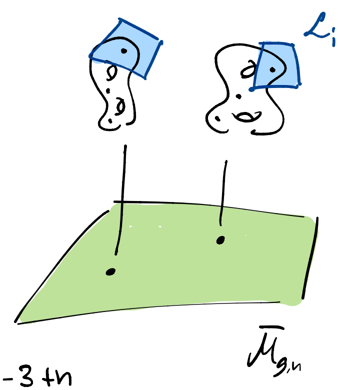
$\dim_{\mathbb{C}} = 3g-3+n$

PROBLEM: Understand the geometry of $\mathcal{M}_{g,n}$ and $\overline{\mathcal{M}}_{g,n}$

E.g. $\chi_{g,n}$ $\quad H^\bullet(\mathcal{M}_{g,n})$ $\quad H^\bullet(\overline{\mathcal{M}}_{g,n})$

Ex [Witten-Kontsevich] $\forall i = 1, \dots, n \rightsquigarrow \mathcal{L}_i \to \overline{\mathcal{M}}_{g,n}$ line bundle

$$[C, p_1, \dots, p_n] \in \overline{\mathcal{M}}_{g,n} \rightsquigarrow \mathcal{L}_i \big|_{[C, p_1, \dots, p_n]} = T^*_{p_i} C$$



$$\psi_i = c_1(\mathcal{L}_i) \in H^2(\overline{\mathcal{M}}_{g,n})$$

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle := \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_1 + \dots + d_n = 3g-3+n$$

Thm $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ satisfy the KdV hierarchy

($\Leftrightarrow$ Virasoro cnstrnts $\Leftrightarrow$ topological recursion)

MOTIVATION:

2d topological quantum gravity $\rightsquigarrow$ $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ correlators

[Witten '91]

$k=0$

2d topological quantum gravity
+
WZW gauge theory at level $k \in \mathbb{N}$ $\rightsquigarrow$ $\langle \tau_{d_1,a_1} \cdots \tau_{d_n,a_n} \rangle$ correlators

[Witten '92]

$$\langle \tau_{d_1,a_1} \cdots \tau_{d_n,a_n} \rangle := \frac{1}{r^{2g}} \int_{\overline{\mathcal{M}}_{g,n}} c_{\mathrm{vtop}}(\mathcal{V})\, \psi_1^{d_1} \cdots \psi_n^{d_n}$$

virtual Euler class / top Chern class

where:
- $r := k+2 \geq 2$
- $\mathcal{V} = \mathcal{V}^r_{g;a_1,\ldots,a_n}$ "$\longrightarrow$" $\overline{\mathcal{M}}_{g,n}$ "vector bundle"

$$\mathcal{V}^*\big|_{[C,p_1,\ldots,p_n]} := H^0\Big(C, K^{1-\frac{1}{r}} \otimes \mathcal{O}\big(\textstyle\sum_i a_i p_i\big)^{\frac{1}{r}}\Big)$$

= roots of holomorphic diffs on $C - \{p_1,\ldots,p_n\}$ w/ prescribed poles @ p_i

IDEA: Take $k \in \mathbb{Z}$ $\Leftrightarrow$ analytic continuation to negative levels

§2) EULER CHARACTERISTIC

$$\boxed{r=-1, \quad a_i=-1}$$

$$\mathcal{V}^*\big|_{[C,p_1,\ldots,p_n]} = H^0\big(C, K^{\otimes 2} \otimes \mathcal{O}(\textstyle\sum_i p_i)\big)$$

$\mathrm{rk}_{\mathbb{C}} = 3g-3+n$

= holom quad diffs on $C - \{p_1,\ldots,p_n\}$ w/ simple poles @ p_i

FACTS: 1) $\mathcal{V} = T_{\mathcal{M}_{g,n}} \to \overline{\mathcal{M}}_{g,n}$ is the (log) tangent bundle

2) (Gauss-Bonnet): $\chi(\mathcal{M}_{g,n}) = \int_{\overline{\mathcal{M}}_{g,n}} c_{top}(T_{\mathcal{M}_{g,n}})$

Q: How to compute?

Thm (from [Chiodo, '08])

$B_{m+1}(x)$ Bernoulli poly

$$c(T_{\mathcal{M}_{g,n}}) = \exp\left[\sum_{m\geq 1} \frac{(-1)^m}{m(m+1)}\left(B_{m+1}(-1)\kappa_m - \sum_{i=1}^{n} B_{m+1}(0)\psi_i^m + B_{m+1}^{(c)}\delta_m\right)\right]$$

known cohomology classes

$B_{m+1}(0) = B_{m+1}$

$B_{m+1}(-1) = B_{m+1} - (-1)^m(m+1)$

$$= \Lambda^\vee \cdot \exp\left(-\sum_{m\geq 1} \frac{\kappa_m}{m}\right)$$

(dual) Hodge class

Corollary.
$$\chi_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \Lambda^\vee \cdot \exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)$$

computable via Hodge integrals manipulations + 2d Toda.

$$\chi_{g,n+1} \overset{!}{=} \int_{\overline{\mathcal{M}}_{g,n+1}} p^*\left(\Lambda^\vee \cdot \exp\left(-\sum_{m\geq 1}\frac{\kappa_m}{m}\right)\right) \underbrace{\exp\left(-\sum_{m\geq 1}\frac{\psi_{n+1}^m}{m}\right)}_{\exp(\log(1-\psi_{n+1})) = 1-\psi_{n+1}}$$

$p: \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$

$\Lambda^\vee = p^*\Lambda^\vee$, $\quad \kappa_m = p^*\kappa_m + \psi_{n+1}$

$$
\vdots
$$

$$
= -\int_{\overline{\mathcal{M}}_{g,n+1}} p^*\left(\Lambda^\vee \cdot \exp\left(-\sum_{m\geq 1} \frac{\kappa_m}{m}\right)\right) \psi_{n+1}
$$

$$
= -\int_{\overline{\mathcal{M}}_{g,n}} \Lambda^\vee \cdot \exp\left(-\sum_{m\geq 1} \frac{\kappa_m}{m}\right) \underbrace{p_* \psi_{n+1}}_{\kappa_0 = 2g-2+n}
$$

$$
= -(2g-2+n)\,\chi_{g,n}
$$

With similar manipulations, HARER-ZAGIER FORMULA

$$
\chi_{g,0} = \sum_{\ell\geq 1} \frac{1}{\ell!} \sum_{\mu_1,\dots,\mu_\ell\geq 1} \int_{\overline{\mathcal{M}}_{g,\ell}} \Lambda^\vee \prod_{i=1}^{\ell} \psi_i^{\mu_i+1} = \frac{1}{2-2g}\zeta(1-2g)
$$

j.w./ D. Lewański, P. Norbury

Toda eqn + ELSV [Dubrovin-Yang-Zagier, '17]

§3) NORBURY's CLASS

$$
\boxed{r = -2, \quad a_i = -1}
$$

$$
\mathcal{V}^*|_{[C,p_1,\dots,p_n]} = H^0\left(C, \left(K \otimes \mathcal{O}\left(\textstyle\sum_i p_i\right)\right)^{1/2}\right)
$$

$rk_{\mathbb{C}} = 2g-2+n$

$= \sqrt{\cdot}$ of holom diffs on $C - \{p_1,\dots,p_n\}$ w/ simple poles @ p_i

$$
\Theta_{g,n} := (-1)^n 2^{g-1} c_{top}(\mathcal{V}) \in H^{2(2g-2+n)}(\overline{\mathcal{M}}_{g,n})
$$

Prop [Norbury]. $\Theta_{g,n}$ is a CohFT satisfying $\Theta_{g,n+1} = \psi_{n+1}\, p^* \Theta_{g,n}$

Conj [Norbury]. $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle^{\Theta} := \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \psi_1^{d_1} \cdots \psi_n^{d_n}$ satisfy the KdV hierarchy (= Brézin-Gross-Witten solution)

Thm [j.w/ N. Chidambaram, E. Garcia-Failde]. Norbury's conj holds true.

STRATEGY:

1) Deform the class: $\Theta^{\varepsilon}_{g,n} = \Theta_{g,n} + \underbrace{\varepsilon\text{-corrections}}_{\text{lower cohom degree}}$, $\varepsilon \in \mathbb{C}$

2) $\Theta^{\varepsilon}_{g,n}$ is a CohFT, semisimple $\forall \varepsilon \neq 0$ ($\Leftarrow$ conj of Kazarian Norbury)

3) $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle^{\Theta^{\varepsilon}}$ are computed by topological recursion $\forall \varepsilon \neq 0$ ($\Leftrightarrow$ unique solution to Virasoro cnstrnts)

4) In the limit $\varepsilon \to 0$, TR still holds

$\Rightarrow$ $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle^{\Theta}$ are computed by TR ($\Leftrightarrow$ unique solution to Virasoro cnstrnts)

the BGW τ-fnct is also a solution [Gross-Newman '92]

§) SPECTRAL CURVES

- For the intersection numbers $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle^{\Theta^{\varepsilon}}$:

$$\begin{cases} x = z^2 - 2\varepsilon z \\ y = z^{-1} \end{cases} \xrightarrow{\varepsilon \to 0} \begin{cases} x = z^2 \\ y = z^{-1} \end{cases}$$

poles of y at ram. pnts

- For the intersection numbs $\int_{\overline{\mathcal{M}}_{g,n}} c(T_{\mathcal{M}_{g,n}})\, \psi_1^{d_1} \cdots \psi_n^{d_n}$

$$\begin{cases} x = z - \log(z) \\ y = z^{-1} \end{cases}$$

Get $\chi_{g,n}$ for $d_1 = \cdots = d_n = 0$.