On the Spin GW/H Correspondence

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§1) INTRO

Fix X smooth proj variety, $\beta \in H_2(X,\mathbb{Z})$

Q: count algebraic curves of X in degree β. [Riemann, Schubert, Hurwitz, ...
Gromov, Witten, ...]

Setup: curves in X are parametrised as $f: C \to X$

$$\overline{\mathcal{M}}_{g,n}(X,\beta) := \left\{ f: C \to X \;\middle|\; \begin{array}{l} C \text{ nodal curve, genus } g,\ n \text{ smooth mrk pnts} \\ f_*[C] = \beta,\ |\mathrm{Aut}(f)| \text{ finite} \end{array} \right\} \Big/ \sim$$

- is proper DM stack
- carries a virtual fund. class [Behrend-Fantechi]

$$[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir} \in A_{\mathrm{vdim}}(-,\mathbb{Q}) \quad w/ \quad \mathrm{vdim} := 3g-3+n+\overbrace{h^0(f^*T_X)}^{\mathrm{Def}(f)} - \underbrace{h^1(f^*T_X)}_{\mathrm{Obs}(f)}$$

$$= \int_\beta c_1(X) + (\dim X - 3)(1-g) + n$$

E.g. | $\beta = 0$, $\overline{\mathcal{M}}_{g,n}(X,0) = \underbrace{\overline{\mathcal{M}}_{g,n} \times X}$

$\dim = (3g-3+n) + \dim X \neq \mathrm{vdim}$ if $\dim X > 0$

GW invariants:

$$\langle \tau_{d_1}(\gamma_1) \cdots \tau_{d_n}(\gamma_n) \rangle^{X,\beta}_{g,n} := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{vir}} \prod_{i=1}^n (ev_i^* \gamma_i)\, \psi_i^{d_i} \in \mathbb{Q}$$

"=" curves of genus g in X
meeting $PD(\gamma_i)$ at ith mrkd pnt w/ tangency condition

where $\gamma_i \in H^\bullet(X,\mathbb{Z})$, $\psi_i = c_1(\mathbb{L}_i)$, $\mathbb{L}_i \to \mathcal{M}_{g,n}(X,\beta)$

$\mathbb{L}_i|_{[f: C \to X]} = T^*_{p_i} C$

Q: how to compute GW invariants of X?

→ A) Toric ⇝ localisation: $(\mathbb{C}^*)^d \curvearrowright X \Rightarrow (\mathbb{C}^*)^d \curvearrowright \overline{\mathcal{M}}_{g,n}(X,\beta)$

[Graber-Pandharipande]

B) Semisimple case ⇝ reconstruction thm

[Givental, Teleman]

→ 0) $\dim X = 0$ ⇝ [Witten, Kontsevich]

1) $\dim X = 1$ ⇝ [Okounkov-Pandharipande] } Virasoro cnstrnts

2) $\dim X = 2$? $\boxed{\text{Today's talk}}$

↘ follows from:

- localisation on $\mathbb{P}^1$
- ELSV formula ⇝ GW/H corresp.
- deg formula

§2) GW of SURFACES

Setup. X smooth proj surface w/ $D \in |K_X|$ smooth connected

⇝ $K_D \overset{\text{adj}}{=} (K_X + D)|_D \cong 2N_{X/D} \Rightarrow (D, \underset{\overset{\shortparallel}{\theta}}{N_{X/D}})$ is a spin curve

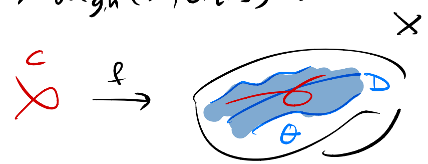
E.g.: A abelian surface, $X = Bl_p A$

Defn. A spin structure on a curve C is a line bundle $\theta \to C$ s.t. $\theta^{\otimes 2} \cong \omega_C$. Then (C,θ) is called a spin curve.

Thm [Lee-Parker, Kiem-Li]. In the above setup, GW invariants of X vanish unless $\beta = d[D]$. Moreover, $\exists\ \overline{\mathcal{M}}_{g,n}(D,d) \overset{\iota}{\hookrightarrow} \overline{\mathcal{M}}_{g,n}(X, d[D])$ and

$$[\overline{\mathcal{M}}_{g,n}(D,d)]^{loc,\theta} \in A_{\text{ldim}}(\mathcal{M}_{g,n}(D,d))$$

↳ $g-1+n+d(1-g(D))$



s.t. $i_* [\overline{\mathcal{M}}_{g,n}(D,d)]^{loc,\theta} = [\overline{\mathcal{M}}_{g,n}(X, d[D])]^{vir}$.

$$\Rightarrow \langle \tau_{d_1}(\gamma_1) \cdots \tau_{d_n}(\gamma_n) \rangle_{g,n}^{X,d[D]} = \int_{[\overline{\mathcal{M}}_{g,n}(D,d)]^{loc,\theta}} \prod_{i=1}^{n} ev_i^* \underbrace{(\gamma_i \cdot [B])}_{\alpha_i} \psi_i^{d_i} =: \langle \tau_{d_1}(\alpha_1) \cdots \tau_{d_n}(\alpha_n) \rangle_{g,n}^{D,\theta,d}$$

spin GW invariants

UPSHOT. 1) We only need to consider $\beta = d[D]$

2) Reduce GW theory of X surfs. to a local theory of (D,θ) spin curves

$\rightsquigarrow$ "dimension reduct" $\rightsquigarrow$ apply OP technique?

suggested by Maulik-Pandharipande

Rmk. Def invariance $\Rightarrow$ spin GW invrnts only dep. on $g(D)$ &

$p(\theta) =: h^0(\theta) \pmod 2$ (parity or Arf invariant)

§3) SPIN GW of $\mathbb{P}^1$

On $\mathbb{P}^1$, $\exists!$ spin structure $\mathcal{O}(-1)$

$$\begin{array}{ccc} \mathcal{C}_{g,n}(\mathbb{P}^1,d) & \xrightarrow{f} & \mathbb{P}^1 \\ \downarrow \pi & & \\ \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d) & & \end{array}$$

Thm [Kiem-Li]

$$[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{loc,\mathcal{O}(-1)} = [\mathcal{M}_{g,n}(\mathbb{P}^1,d)]^{vir} \frown c_{g-1+d}(R^1\pi_* f^*\mathcal{O}(-1))$$

$= $ Euler class

$$\overset{\frown}{A}_{g-1+n+d}(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d))$$

UPSHOT: total control on localised fund. class + localisation.

$\Downarrow$

Thm [GKLS]. The spin GW invrnts of equivariant $(\mathbb{P}^1, \mathcal{O}(-1))$

$$\int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{loc,\mathcal{O}(-1)}} \left(\prod_{i=1}^{n} \frac{ev_i^*(0)}{1 - z_i \psi_i} \right) \left(\prod_{j=1}^{m} \frac{ev_{n+j}^*(\infty)}{1 - z_{n+j}\psi_{n+j}} \right)$$

are expressed in terms of double Hodge integrals

$$\int_{\overline{\mathcal{M}}_{g,\ell}} \frac{\Lambda(2)\Lambda(-1)}{\prod_{k=1}^{\ell}(1 - z_k\psi_k)} \quad \text{w/} \quad \Lambda(t) = \sum_{k=0}^{g} t^k \lambda_k \quad \text{Chern poly of } \mathbb{E} \to \overline{\mathcal{M}}_{g,n}$$

Hodge bundle.

Q. How to compute DHIs?

§4) SPIN HHs

Spin HHs: signed enumeration of branched covers of curves [Eskin-Okounkov-Pandharipande] Gunningham, ...

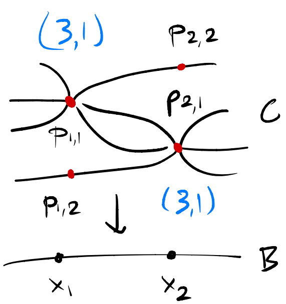
$\downarrow$

parity of spin structure

Defn. Fix (B,θ) a spin curve, $x_1,\dots,x_m \in B$, $\mu^1,\dots,\mu^m \vdash d$ ODD

all parts are odd integers

If $f: C \to B$ is a $d:1$ cover branched at x_i w/ ram. profile μ^i



$$\rightsquigarrow \quad \mathrm{Ram}(f) := \sum_{j=1}^{m} \sum_{k=1}^{\ell(\mu^j)} \underbrace{(\mu^j_k - 1)}_{\text{EVEN}} p_{j,k}$$

$$\rightsquigarrow \quad \theta_f := f^*\theta \otimes \mathcal{O}\left(\tfrac{1}{2}\mathrm{Ram}(f)\right)$$

$\hookrightarrow$ is a spin structure on C

Defn

$$\mathrm{Hur}_d(B,\theta;\mu^1,\dots,\mu^m) := \sum_{[f:C\to B]} \frac{(-1)^{h^0(\theta_f)}}{|\mathrm{Aut}(f)|}$$

Simplest case: $(B,\theta) = (\mathbb{P}^1, \mathcal{O}(-1))$, $\mu^1 = \mu$, $\mu^2,\dots,\mu^m = (3,1,\dots,1)$

Thm [GKL, Alexandrov-Shadrin]

$(3,1,\dots,1)$ + loc. explicit

$$\mathrm{Hur}_d(\mathbb{P}^1,\mathcal{O}(-1);\mu,((3,1,\dots,1))^b) = \# \int_{\overline{\mathcal{M}}_{g,\ell(\mu)}} \frac{\Lambda(2)\Lambda(-1)}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}$$

$\hookrightarrow 2b = 2g-2+\ell(\mu)+d$

UPSHOT: spin HHs $\xleftrightarrow[\text{formula}]{\text{ELSV-type}}$ DHIs $\xleftrightarrow{\text{localis.}}$ spin GW of $(\mathbb{P}^1, \mathcal{O}(-1))$

spin HHs $\rightsquigarrow$ integrable hierarchy of BKP type

$\Leftrightarrow$ Fock space formalism

spin HHs $\longleftrightarrow$ spin GW of $(\mathbb{P}^1, \mathcal{O}(-1))$: spin GW/H correspondence (for equivariant $(\mathbb{P}^1, \mathcal{O}(-1))$)

Thm. [GKLS] Generating series of spin GW invariants of equiv. $(\mathbb{P}^1, \mathcal{O}(-1))$ is expressed as a vacuum expectation value on Fock space.

$$\tau(\underbrace{t_0, t_1, \dots}_{\text{insert } 0}; \underbrace{s_0, s_1, \dots}_{\text{insert } \infty}; \overset{\text{genus var}}{u}, \underset{\text{deg var.}}{q}) = \left\langle 0 \right| e^{\sum_i t_i B_i} e^{\alpha_1} \left(\frac{q}{u}\right)^H e^{\alpha_{-1}} e^{\sum_j s_j B_j^*} \left| 0 \right\rangle$$

$$B_i, \alpha_1, \dots \in \mathcal{D}(\infty)$$

Thus, it is a τ-fnct of the 2-BKP hierarchy.

B: type B; see Paolo's talk

$\Longrightarrow$ ALGORITHM to compute spin GW invariants of $(\mathbb{P}^1, \mathcal{O}(-1))$

Corollary.

$$\left\langle \tau_{d_1}([pt]) \cdots \tau_{d_n}([pt]) \right\rangle_{g,n}^{\bullet, \mathbb{P}^1, \mathcal{O}(-1), d=1} = \prod_{i=1}^{n} \frac{d_i!}{(2d_i+1)!} (-2)^{-d_i}$$

$$\left\langle \quad " \quad \text{———} \quad " \quad \right\rangle^{" \, - \, ", d=2} = \frac{1}{2} \prod_{i=1}^{n} \frac{2\, d_i!}{(2d_i+1)!} (-2)^{d_i}$$

conj by Maulik-Pandharipande, proved by Kiem-Li & Kool-Thomas

$$\left\langle \quad " \quad \text{———} \quad " \quad \right\rangle^{" \, - \, ", d=3} = \text{much more complicated}$$

Corollary. Recursive formula for 1-pnt invrnt ($n=1$). E.g.

$$\left\langle \tau_{10} \right\rangle_{6,1}^{\bullet, \mathbb{P}^1, \mathcal{O}(-1), 4} = \frac{184\,123\,634\,063}{43\,251\,591\,256\,473\,600} \qquad (\simeq 0.04\ s)$$

For future work:

A) Full spin GW/H corresp. follows from conj degeneration formula

↳ general spin target (D, θ)

B) Virasoro conj for GW of surfaces (w/ smooth canonical)