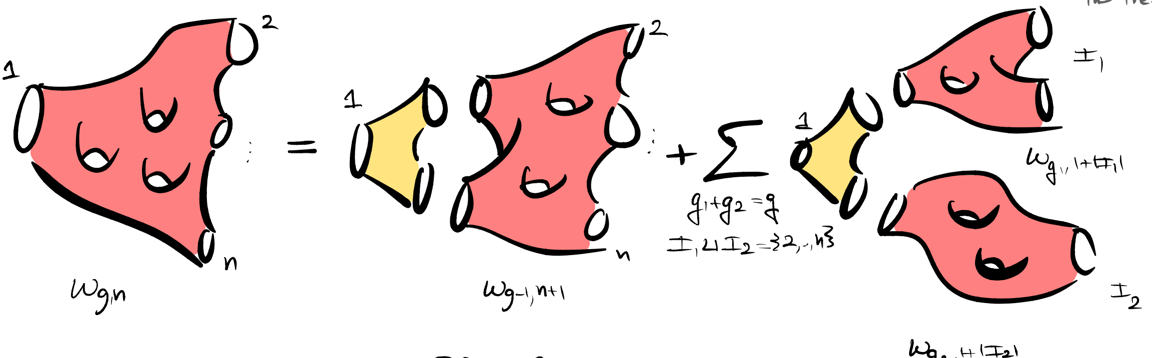
TOPOLOGICAL RECURSION

ICTP: Physics Latam – Mathematics & High Energy Physics

REFS: B. Eynard, "A short overview of the Topological Recursion", math-ph/1412.3286
G. Borot, "Lecture notes on topological recursion and geometry", math-ph/1705.09986
A. Giacchetto, "Geometric and topological recursion and invariants of the moduli space of curves" PhD thesis



spectral curve INPUT $\xrightarrow{\text{TOPOLOG. REC.}}$ OUTPUT correlators

$\mathcal{S} = (\Sigma, \dots)$

Riemann surface + extra data

$\hookrightarrow$ $(\omega_{g,n}(\mathcal{S}))_{g,n\geq 0}$ of n-form on Σ^n

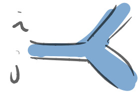
for $n=0$, $\omega_{g,0} = F_g \in \mathbb{C}$, free energies

WHY? many applications: RMT, volumes of moduli spaces, mirror symmetry, GW invariants / topological string, Hurwitz numb, ..

many properties: Virasoro cnstrnts, integrability, SW geometry, ...

§1) ORIGIN: RMT

- '50s: Wigner → energy spectrum of heavy nuclei
  $\rightsquigarrow$ system with complicated Hamiltonians
- '70s: 't Hooft → QCD w/ N_c = # colourings → ∞

$\rightsquigarrow$ connections w/ graphs on surfaces 

- '80s: Brézin - Itzykson - Zuber $\to$ 2d quantum gravity
- '04 - '07: Chekhov - Eynard - Orantin $\to$ TR

Setup: $\mathcal{H}_N = \{ N\times N \text{ Herm. matrices} \}$

$$d\mu(M) = e^{-\frac{1}{\hbar}\mathrm{Tr}\, V(M)}\, dM \qquad \rightsquigarrow \qquad Z = \int_{\mathcal{H}_N} d\mu(M)$$

$\uparrow$ U(N)-invariant Lebesgue measure

Natural observables: $\langle f \rangle = \mathbb{E}[f] = \frac{1}{Z}\int_{\mathcal{H}_N} f(M)\, d\mu(M)$

- Traces: $(\mathrm{Tr}\, M) \quad \rightsquigarrow \quad \sum_{k\geq 0} \langle \mathrm{Tr}\, M^k \rangle\, x^{-k-1} = \left\langle \mathrm{Tr}\, \frac{1}{x-M} \right\rangle$

  resolvent

- Charact. poly: $\langle \det(x-M) \rangle = \left\langle e^{\mathrm{Tr}\log(x-M)} \right\rangle$

$$= \left\langle e^{\int_\infty^x \mathrm{Tr}\left(\frac{1}{x'-M}\right) dx'} \right\rangle$$

$$= e^{\sum_n \frac{1}{n!} \int_\infty^x \cdots \int_\infty^x \left\langle \prod_{i=1}^n \mathrm{Tr}\left(\frac{1}{x_i - M}\right)\right\rangle_c dx_1 \cdots dx_n}$$

$\uparrow$ cumulants

- Spectral density: $\rho(x) = \left\langle \frac{1}{N} \sum_{i=1}^n \delta(x-\lambda_i) \right\rangle$

  $\uparrow$ determined by resolvent

$$\delta(x-\lambda) = \frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{x-\lambda-i\varepsilon}\right)$$

Notation: $W_n(x_1, \dots, x_n) = \left\langle \prod_{i=1}^n \mathrm{Tr}\left(\frac{1}{x_i - M}\right) \right\rangle_c$ n-correlators

Q: How to compute W_n?

Idea: compute W_n as $N \to \infty$ w/ $t = \hbar N = \text{cnst}$

↑ 't Hooft parameter

$$\rightsquigarrow \quad Z = \sum_{g \geq 0} \hbar^{2g-2} F_g$$

↑ Feynman diagram expns : $-(V - E + F) = 2g-2$

$$W_n = \sum_{g \geq 0} \hbar^{2g-2+n} W_{g,n}$$

Eynard and collab, '04: $\boxed{(W_{g,n})_{g,n} \text{ are computed by a UNIVERSAL RECURSIVE FORMULA, depends on } W_{0,1}}$

in $2g-2+n$

LOOP EQNs

leading term of resolvent ↔ leading term of spectr. dens.

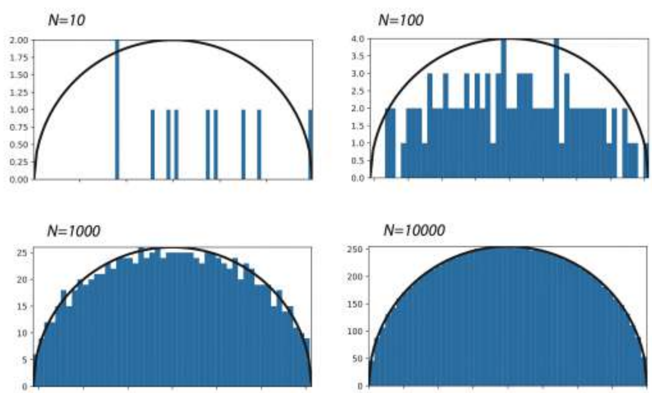
SPECTRAL CURVE

Example: GUE. Take $V(M) = \frac{M^2}{2}$

$$\rightsquigarrow \quad \rho(x) = \frac{1}{\hbar} \rho_0(x) + \cdots$$

$$= \frac{1}{2\pi} \sqrt{4 - x^2}$$

Wigner semicircle law →



Interactive: bertrand.eynard.free.fr/RMW1.html

Credit: Lauren H. Li

Eynard-Orantin '07: TOPOLOGICAL RECURSION as a general theory w/o matrix model

§2) JT GRAVITY

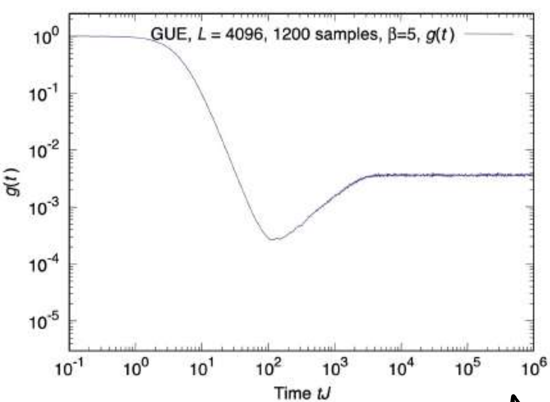
$\sim$ 2015: Kitaev noticed:

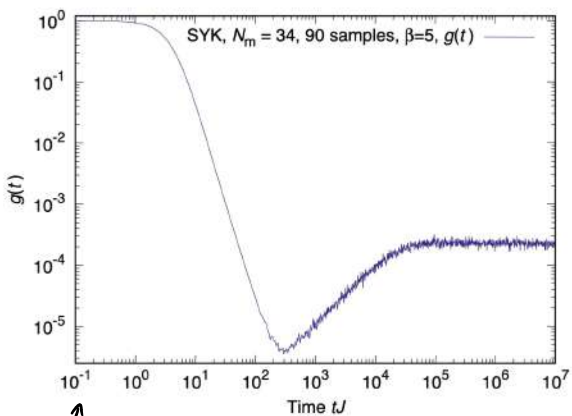
$$\boxed{\begin{array}{ccc} \text{SYK model} & \longleftrightarrow & \text{black hole} \\ \text{QM} & & \text{gravity} \end{array}}$$

$\rightarrow$ gravitational dual: JT gravity

$\rightarrow$ behaves like a matrix model w/ $\rho_0(x) = \frac{1}{(2\pi)^2} \sinh(2\pi\sqrt{-x})$

$\rightsquigarrow$ apply MM techiques to JT gravity?





Credit: Jordan J. COLTER et. al.

some spectral form factor

The action describing JT gravity is

$$S_{JT}(g,\phi) = -S_0\,\chi(M) - \frac{1}{2}\int_M d^2x \sqrt{g}\, \Phi\,(R+2) - \text{brdry terms}$$

Riemannian metric on 2d mnfld M

scalar field dilaton

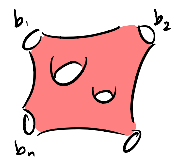
$\Downarrow$

$R = -2 \Rightarrow$ hyperbolic metric

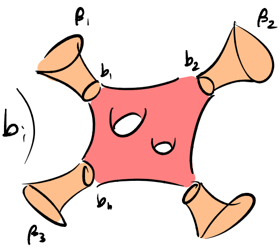
Thus, the path integral involves $\int_{\substack{\text{hyperbolic}\\ \text{metrics}}}$. For instance, a natural correlation fncts is

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle_c = \sum_{g=0}^{\infty}\left(e^{-S_0}\right)^{2g-2+n} Z_{g,n}(\beta_1,\dots,\beta_n)$$

$$= \sum_{g=0}^{\infty}\left(e^{-S_0}\right)^{2g-2+n}\int_0^{+\infty}\left(\prod_{i=1}^{n} db_i\, b_i\right) \underbrace{Z^{\text{trump}}(\beta_i,b_i)}_{\text{hyperbolic trumpet}}\ \underbrace{V_{g,n}(b_1,\dots,b_n)}_{\text{Weil-Petersson volumes}}$$

- $$Z^{\text{trump}}(\beta,b) = \frac{e^{-\frac{b^2}{4\beta}}}{\sqrt{4\pi\beta}}$$
- $V_{g,n}(b_1,\dots,b_n) = \mathrm{Vol}\Bigg(\underbrace{\Big\{$  hyperbolic metrics on surface of genus g, n boundaries of length $b_1,\dots,b_n \Big\}/\sim}_{\mathcal{M}^{hyp}_{g,n}(b_1,\dots,b_n)\ =\ \text{moduli space of hyp. Riemann surfaces}} \Bigg)$

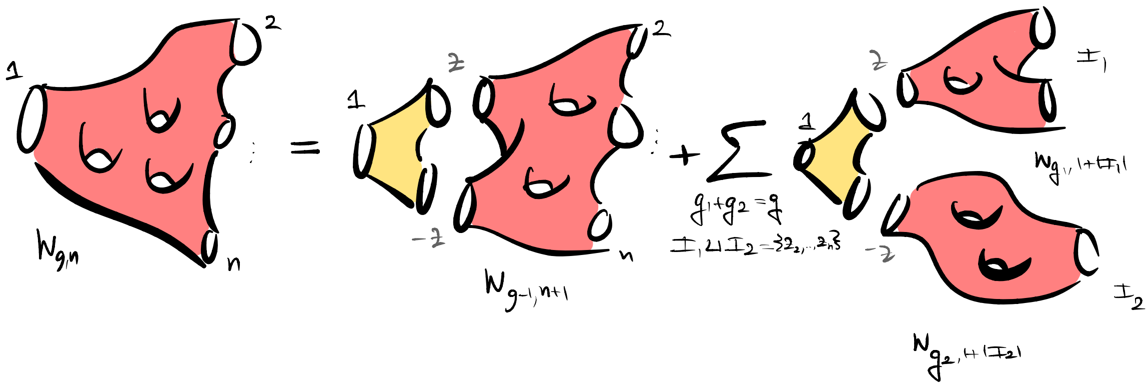
$$V_{g,n}(b_1,\dots,b_n) = \int_{\mathcal{M}^{hyp}_{g,n}(b_1,\dots,b_n)} d\mu_{WP}$$

$$\langle Z(\beta_1)\cdots Z(\beta_n)\rangle_c = \sum_{g=0}^{\infty}\left(e^{-S_0}\right)^{2g-2+n}\int_0^{+\infty}\left(\prod_{i=1}^{n} db_i\, b_i\right)$$ 

Since JT $\leftrightarrow$ MM, we expect to be able to compute $V_{g,n}$ recursively!

Set $$W_{g,n}(z_1,\dots,z_n) = \mathcal{L}[V_{g,n}](z_1,\dots,z_n) \stackrel{!}{=} \int_0^{\infty}\left(\prod_{i=1}^{n} db_i\, b_i\, e^{-z_i b_i}\right) V_{g,n}(b_1,\dots,b_n)$$

TOPOLOGICAL RECURSION FOR WP volumes / JT gravity



$$\boxed{W_{g,n}(z_1,\dots,z_n) = \operatorname*{Res}_{z\to 0} \frac{dz}{z_1^2-z^2}\frac{-\pi}{\sin 2\pi z}\left(W_{g-1,n+1}(z,-z,z_2,\dots,z_n) + \sum_{\substack{g_1+g_2=g \\ I_1\sqcup I_2=\{z_2,\dots,z_n\}}}^{\text{no }(0,1)} W_{g_1,1+|I_1|}(z,I_1)\, W_{g_2,1+|I_2|}(-z,I_2)\right)}$$

w/ $W_{0,2}(z_1,z_2) = \dfrac{1}{(z_1-z_2)^2}$.

Eynard-Orantin '07
after Mirzakhani '07

Example: $(1,1)$

$$\begin{aligned}
W_{1,1}(z_1) &= \operatorname*{Res}_{z\to 0} \frac{dz}{z_1^2-z^2}\frac{-\pi}{\sin 2\pi z}\left(W_{0,2}(z,-z)\right) \\
&= \operatorname*{Res}_{z\to 0} \frac{dz}{z_1^2-z^2}\frac{-\pi}{\sin 2\pi z}\frac{1}{(2z)^2} \\
&= \operatorname*{Res}_{z\to 0} \frac{dz}{z_1^2}\left(1+\left(\frac{z}{z_1}\right)^2+O(z^4)\right)\frac{-\pi}{2\pi z-\frac{1}{6}(2\pi z)^3+O(z^5)}\frac{1}{(2z)^2} \\
&= \frac{-1}{8z_1^2}\operatorname*{Res}_{z\to 0}\frac{dz}{z^3}\left(1+\left(\frac{z}{z_1}\right)^2+O(z^4)\right)\left(1+\frac{2\pi^2}{3}z^2+O(z^4)\right) \\
&= \frac{-1}{8z_1^2}\left(\frac{1}{z_1^2}+\frac{2\pi^2}{3}\right)
\end{aligned}$$

MATHEMATICA code @ github.com/agiacche

Why TR ? $\longrightarrow$ Mirzakhani's recursion '07
in this case   matrix model $\leftrightarrow$ SYK $\leftrightarrow$ JT

§3) HOW TO GENERALIZE?

$$\omega_{g,n}(z_1,\dots,z_n) = \operatorname*{Res}_{z\to a} \frac{1}{2}\int_{\sigma(z)}^{z} \omega_{0,2}(z_1,\cdot)\, \frac{1}{\omega_{0,1}(z)-\omega_{0,1}(\sigma(z))}\Big(\omega_{g-1,n+1}(z,\sigma(z),z_2,\dots,z_n) + \sum^{\text{no }(0,1)}_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=\{z_2,\dots,z_n\}}} \omega_{g_1,1+I_1}(z,I_1)\,\omega_{g_2,1+I_2}(\sigma(z),I_2)\Big)$$

$$\omega_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2}$$

1) $\omega_{g,n}(z_1,\dots,z_n) = W_{g,n}(z_1,\dots,z_n)\,dz_1\cdots dz_n$
$\hookrightarrow$ symm n-form on Σ^n, $\Sigma=\mathbb{C}$

2) $\frac{1}{2}\int_{-z}^{z} \omega_{0,2}(z_1,\cdot) = \frac{dz_1}{2}\int_{z'=-z}^{z} \frac{dz'}{(z_1-z')^2} = \frac{z\,dz_1}{z_1^2-z^2}$

3) Introduce the fncts
$$\begin{cases} x(z) = \frac{z^2}{2} \\ y(z) = \frac{\sin(2\pi z)}{2\pi} \end{cases}$$

Observe: ► $dx = z\,dz$ vanishes @ $z=0=a$

► $x(-z)=x(z)$ $\rightsquigarrow$ $\sigma: z\mapsto -z$ involution, exchange sheets of, leaves $a=0$ invariant

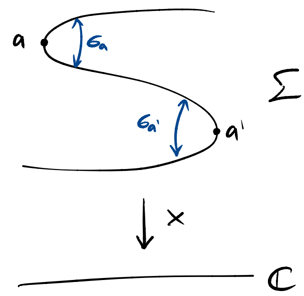
4) $\omega_{0,1}(z) = y(z)\,dx(z) = z\,.\,\frac{\sin(2\pi z)}{2\pi}\,dz$ $\rightsquigarrow$ $\omega_{0,1}(z)-\omega_{0,1}(\sigma(z)) = \frac{z\sin(2\pi z)\,dz}{\pi}$

Definition. A spectral curve is the data $\mathcal{S}=(\Sigma, x, \omega_{0,1}, \omega_{0,2})$ where

- Σ is a Riemann surface (not necessarily cmct)
- $x : \Sigma \to \mathbb{C}$ meromorphic s.t. dx has finitely simple zeros $a \in R$

$$\Rightarrow \exists \text{ local involution } \sigma_a : U_a \to U_a \text{ w/}$$
$$x(\sigma_a(z)) = x(z), \quad \sigma_a(a) = a, \quad \sigma_a \neq id$$

- $\omega_{0,1}(z)$ meromorphic 1-form on Σ
  (often, $\omega_{0,1} = y\,dx$, $\Sigma = \{P(x,y)=0\}$)
- $\omega_{0,2}(z_1,z_2)$ symm meromorphic 2-form on Σ^2
  w/ double pole along diagonal



For $g \geq 0$, $n \geq 1$, $2g-2+n>0$, DEFINE $\omega_{g,n}$ recursively as

$$\boxed{\omega_{g,n}(z_1,\dots,z_n) = \sum_{a\in R} \operatorname*{Res}_{z\to a} \frac{\frac{1}{2}\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_1,\cdot)}{\omega_{0,1}(z)-\omega_{0,1}(\sigma_a(z))} \Big(\omega_{g-1,n+1}(z,\sigma_a(z),z_2,\dots,z_n) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{z_2,\dots,z_n\}}}^{\text{no }(0,1)} \omega_{g_1,1+I_1}(z,I_1)\,\omega_{g_2,1+I_2}(\sigma_a(z),I_2) \Big)}$$

TOPOLOGICAL REC. FORMULA

§ 3.1) PROPERTIES & EXAMPLES

i) [SYMMETRY] $\omega_{g,n}$ is symmetric in $z_1,\dots,z_n$

ii) [POLE STRCTR] $\omega_{g,n}$ is a meromorphic, w/ poles only at ramification pnts of order $\leq 2(3g-3+n)+2$ and no residue: $\operatorname*{Res}_{z\to a} \omega_{g,n+1}(z_1,\dots,z_n,z) = 0.$

iii) [DILATON EQN]. Take $\Phi_{0,1}(z)$ s.t. $d\Phi_{0,1} = \omega_{0,1}$. Then

$$\omega_{g,n}(z_1,\dots,z_n) = \frac{1}{2g-2+n} \sum_{a\in R} \operatorname*{Res}_{z\to a} \omega_{g,n+1}(z_1,\dots,z_n,z)\, \Phi_{0,1}(z)$$

$\rightsquigarrow$ define $\omega_{g,0} = F_g = \frac{1}{2g-2} \sum_{a\in R} \operatorname*{Res}_{z\to a} \omega_{g,1}(z)\, \Phi_{0,1}(z)$

iv) [HOMOGENEITY] $S = (\Sigma, x, \omega_{0,1}, \omega_{0,2})$, $S' = (\Sigma, x, \lambda\omega_{0,1}, \omega_{0,2})$ $\quad \lambda \in \mathbb{C}^*$

$\rightsquigarrow$ $\omega_{g,n}(S') = \lambda^{-(2g-2+n)}\, \omega_{g,n}(S)$

v) [DEFORMATION EQNS]. If S depens on parameters $(t_1, t_2, \dots)$

$\rightsquigarrow$ $\frac{\partial}{\partial t_i} \omega_{g,n} = \dots$

vi) [QUANTUM CURVE] If $\Sigma = \{P(x,y)=0\} \subseteq \mathbb{C}\times\mathbb{C}$, then

$$\psi(x,\hbar) = \exp\left(\sum_{g,n} \frac{\hbar^{2g-2+n}}{n!} \int^x \cdots \int^x \omega_{g,n} \right)$$

is the WKB solution of a quantization of P:

$$\hat{P}(x, \hbar \tfrac{d}{dx})\psi = 0$$

a) [AIRY] $\Sigma = \mathbb{C}$, $x(z) = \frac{z^2}{2}$, $y(z) = z$, $\omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}$

$\rightsquigarrow$ 2d topological gravity / Kontsevich MM.

b) [BESSEL] $\Sigma = \mathbb{C}$, $x(z) = \frac{z^2}{2}$, $y(z) = z^{-1}$, $\omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1-z_2)^2}$

$\rightsquigarrow$ Brezin-Gross-Witten MM

c) [WP VOLUMES / JT / SYK]

$$\Sigma = \mathbb{C}, \quad x(z) = \frac{z^2}{2}, \quad y(z) = \frac{\sin(2\pi z)}{2\pi} \quad \omega_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2}$$

SUPER $\rightsquigarrow$ " ——— " $\quad y(z) = \dfrac{\cos(2\pi z)}{2\pi z}$ " ——— "

d) [COUNTING of GEODESICS of length $\leq L$ / MASUR-VEECH VOLUMES]

$$\Sigma = \mathbb{C}, \quad x(z) = \frac{z^2}{2}, \quad y(z) = \frac{\sin(2\pi z)}{2\pi}, \quad \omega_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{2(z_1-z_2)^2}\left(1 + \frac{1}{\mathrm{sinc}^2\left(\frac{z_1-z_2}{t}2\pi\right)}\right)$$

Laplace dual to L

e) [HURWITZ NUMBERS]

$$\Sigma = \mathbb{C}, \quad x(z) = \log(z) - z, \quad y(z) = z \quad \omega_{0,2}(z_1,z_2) = \frac{dz_1\,dz_2}{(z_1-z_2)^2}$$

f) [PAINLEVÉ I]

$$\Sigma = \mathbb{C}/\mathbb{Z}\oplus\tau\mathbb{Z}, \quad x(z) = \wp(z,\tau), \quad y(z) = \wp'(z,\tau), \quad \omega_{0,2}(z_1,z_2) = \left(\wp(z_1-z_2,\tau) + \frac{\pi}{\mathrm{Im}\,\tau}\right)dz_1\,dz_2$$

g) [MATRIX MODELS] $\quad \Sigma = \{y = 2\pi i\,\rho_0(x)\}, \quad \omega_{0,2}$ = canonical

h) [GW of TORIC CY3] $\quad \Sigma = \{H(e^x, e^y) = 0\}, \quad \omega_{0,2}$ = canonical

$\uparrow$ mirror curve

i) [JONES POLY] $\quad \Sigma = \{A(e^x, e^y) = 0\}, \quad \omega_{0,2}$ = canonical

(conjectural)

$\uparrow$ A-poly character variety

§4) CONNECTION w/ MODULI SPACES

SYK / JT correlators

TOPOLOGICAL REC. correlators on $y = \frac{\sin(2\pi\sqrt{x})}{2\pi}$

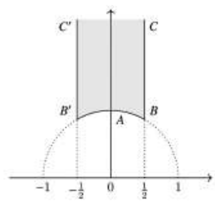
volumes MODULI of hyperbolic RIEMMAN SURFACES

$$\boxed{\mathcal{M}_{g,n} = \left\{ \begin{array}{c} \text{cmlx structures} \\ \text{on surfaces of genus } g \\ \text{w/ } p_1, \dots, p_n \text{ mrk pnts} \end{array} \right\} \Big/ \sim}$$

$\leftarrow$ moduli space of (complex) curves

E.g.

$$\mathcal{M}_{1,1} = \left\{ \begin{array}{c} \text{elliptic} \\ \text{curves} \end{array} \right\} = \mathbb{H} / SL(2,\mathbb{C}) =$$



$\ni \tau \mapsto \mathbb{C} / \mathbb{Z} \oplus \tau\mathbb{Z}$

Uniformization thm (Riemann). $\forall\, b_1, \dots, b_n \in [0, +\infty)$

$$\mathcal{M}_{g,n} \cong \mathcal{M}^{hyp}_{g,n}(b_1, \dots, b_n)$$

$\rightsquigarrow$ every topological invariant of $\mathcal{M}^{hyp}_{g,n}(b_1, \dots, b_n)$ can be expressed as a topological invariant of $\mathcal{M}_{g,n}$

E.g. (Wolpert - Mirzakhani)

$$V_{g,n}(b_1, \dots, b_n) = \int_{\mathcal{M}^{hyp}_{g,n}(b_1, \dots, b_n)} d\mu_{WP} = \int_{\mathcal{M}_{g,n}} \exp\left(2\pi^2 \kappa_1 + \sum_{i=1}^{n} \frac{b_i^2}{2} \psi_i \right)$$

where $\kappa_1, \psi_1, \dots, \psi_n \in H^2(\mathcal{M}_{g,n})$ are natural cohomology classes.

For instance,

$$\psi_i = c_1(\mathcal{L}_i), \qquad \mathcal{L}_i \to \mathcal{M}_{g,n} \text{ line bundle with } \mathcal{L}_i|_{C,p_1,\dots,p_n} = T^*_{p_i}C$$

TOPOLOGICAL REC. correlators on a general spectral curve $\longleftrightarrow$ intersection numbers on MODULI SPACE of COMPLEX CURVES

Can this correspondence be generalised to arbitrary spectral curves?

Thm (Eynard/Dunin-Barkowski-Orantin-Shadrin-Spitz). Fix $\mathcal{S} = (\Sigma, x, \omega_{0,1}, \omega_{0,2})$.
For simplicity, assume x has only one simple ram. pnt a.

$\rightsquigarrow$ local coord ζ around a s.t. $x(z) - x(a) = \frac{\zeta^2(z)}{2}$

- [DIFFERENTIALS]

$$\xi(z) = \int^z \frac{\omega_{0,2}(z_0,\cdot)}{d\zeta(z_0)}\Big|_{z_0=a} \rightsquigarrow d\xi_k(z) = d\left(\frac{d^k}{dx^k}\xi(z)\right)$$

- [TOPOLOGICAL FIELD THEORY]: $y_0 \in \mathbb{C}$

$$y_0 = \frac{dy(z)}{d\zeta(z)}\Big|_{z=a} \qquad W_{g,n}$$

- [TRANSLATION] $T(u) = u\left(1 - \exp\left(-\sum_{m=1}^{\infty} t_m u^m\right)\right) \in u^2 \cdot \mathbb{C}[[u]]$

$$T(u) = u y_0 + \frac{1}{\sqrt{2\pi u}} \int_{\text{steepest descent}} e^{\frac{\zeta^2(z)}{2u}} \omega_{0,1}(z)$$

- [ROTATION] $R(u) = \exp\left(\sum_{m=1}^{\infty} r_m u^m\right) \in 1 + u\cdot\mathbb{C}[\![u]\!]$

$$R(u) = \frac{1}{\sqrt{2\pi u}} \int_{\text{steepest descent}} e^{\frac{\zeta^2(z)}{2u}}\, \xi(z)\, dx(z)$$

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$$\Rightarrow \boxed{\omega_{g,n}(z_1,\dots,z_n) = \int_{\overline{\mathcal{M}}_{g,n}} \Omega_{g,n}(\mathcal{S}) \prod_{i=1}^{n} \sum_{k_i \geq 0} \psi_i^{k_i}\, d\xi^{k_i}(z_i)}$$

$$\Omega_{g,n}(\mathcal{S}) = y_0^{-(2g-2+n)} \exp\left(\sum_{m=1}^{\infty}\left(t_m \kappa_m - r_m\left(\sum_{i=1}^{n} \psi_i^m - \delta_m\right)\right)\right)$$

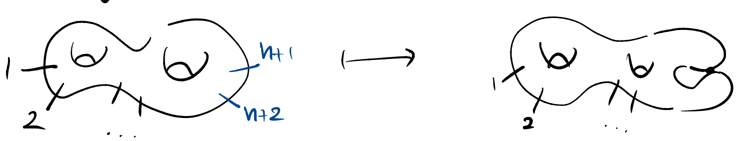
$\in H^\bullet(\overline{\mathcal{M}}_{g,n})$

natural cohomology classes on $\overline{\mathcal{M}}_{g,n}$

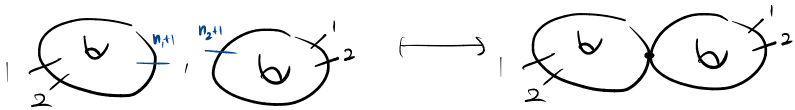
1) GAUGE/STRING duality [Gopakumar-Mazenc]

2) $\Omega_{g,n}$ is called "COHOMOLOGICAL FIELD THEORY", a cohomolgical generalisation of 2d topological field theory

$$\overline{\mathcal{M}}_{g-1,n+2} \xrightarrow{q} \overline{\mathcal{M}}_{g,n}$$



$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \xrightarrow{r} \overline{\mathcal{M}}_{g,n}$$



$g_1 + g_2 = g$

$n_1 + n_2 = n$

$\rightsquigarrow$ CohFT means

$$\left.\begin{aligned} q^* \Omega_{g,n} &= \Omega_{g-1,n+2} \\ r^* \Omega_{g,n} &= \Omega_{g_1,n_1+1} \otimes \Omega_{g_2,n_2+1} \end{aligned}\right\} \text{"cohomological locality axioms"}$$

Examples.

- $\mathcal{S}^{\mathrm{Airy}} = \left(\mathbb{C},\ x(z) = \frac{z^2}{2},\ y(z) = z,\ \omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right)$

$\Rightarrow$ $\Omega_{g,n}(\mathcal{S}^{\mathrm{Airy}}) = 1$

$\Rightarrow$ $\omega_{g,n}^{\mathrm{Airy}}$ compute ψ-class int. numbers
[Witten's conj / Kontsevich thm]

- $\mathcal{S}^{\mathrm{WP}} = \left(\mathbb{C},\ x(z) = \frac{z^2}{2},\ y(z) = \frac{\sin(2\pi z)}{2\pi},\ \omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right)$

$\Rightarrow$ $\Omega_{g,n}(\mathcal{S}^{\mathrm{WP}}) = \exp(2\pi^2 \kappa_1)$

$\Rightarrow$ $\omega_{g,n}^{\mathrm{WP}}$ compute WP int. numbers
[Mirzakhani's recursion]

- $\mathcal{S}^{\mathrm{Hur}} = \left(\mathbb{C},\ x(z) = \log(z) - z,\ y(z) = z,\ \omega_{0,2}(z_1,z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}\right)$

$\Rightarrow$ $\Omega_{g,n}(\mathcal{S}^{\mathrm{Hur}}) = \Lambda^* =$ Hodge class $\Rightarrow$ [ELSV formula]