Probability and Geometry in, on and of non-Euclidian spaces

October 2-6, 2023

Resurgent large genus asymptotics of intersection numbers

j/w B. Eynard, E. Garcia-Failde, P. Gregori, D. Lewański

arXiv: AG/2309.03143

Alessandro Giacchetto

Université Paris-Saclay $\rightarrow$ ETH Zurich

A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + O(m^{-1})\right)$

Con: asymptotically exact
Pro: closed-form

A case study: $m!$

Enumerative problem: $c_m = \#\left\{\begin{array}{c}\text{arrangements of } m \text{ distinct objects}\\ \text{into } m \text{ distinct boxes}\end{array}\right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m\left(1 + O(m^{-1})\right)$

Con: asymptotically exact
Pro: closed-form

A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact

Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m\left(1 + O(m^{-1})\right)$

Con: asymptotically exact

Pro: closed-form

A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + O(m^{-1})\right)$

Con: asymptotically exact
Pro: closed-form

A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \mathrm{O}(m^{-1})\right)$

Con: asymptotically exact
Pro: closed-form

A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \frac{1}{12}m^{-1} + O(m^{-2})\right)$

Con: asymptotically exact
Pro: closed-form

A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

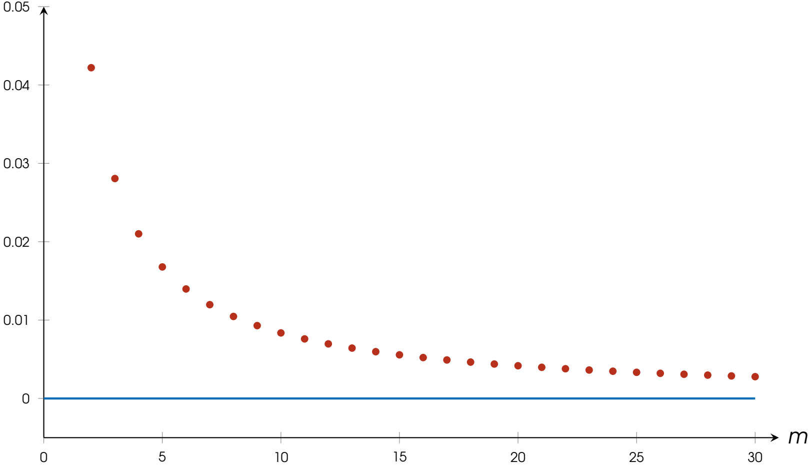
Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \frac{1}{12} m^{-1} + \frac{1}{288} m^{-2} + O\left(m^{-3}\right)\right)$

Con: asymptotically exact
Pro: closed-form

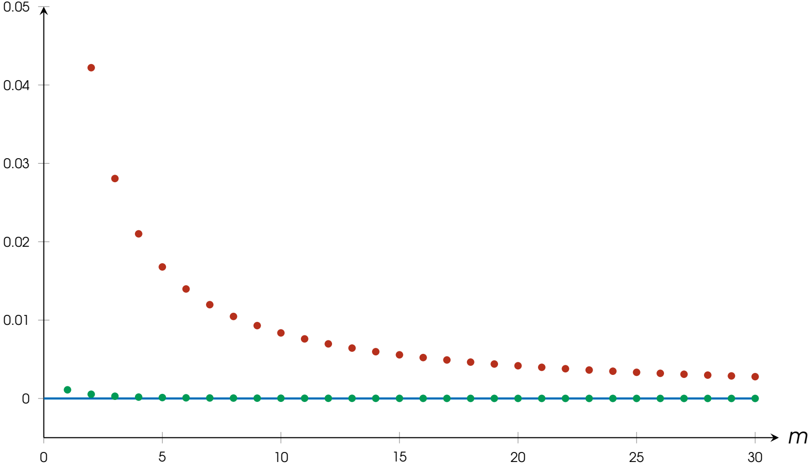
Visualising Stirling's formula

$$\frac{m!}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} - 1 = O(m^{-1})$$



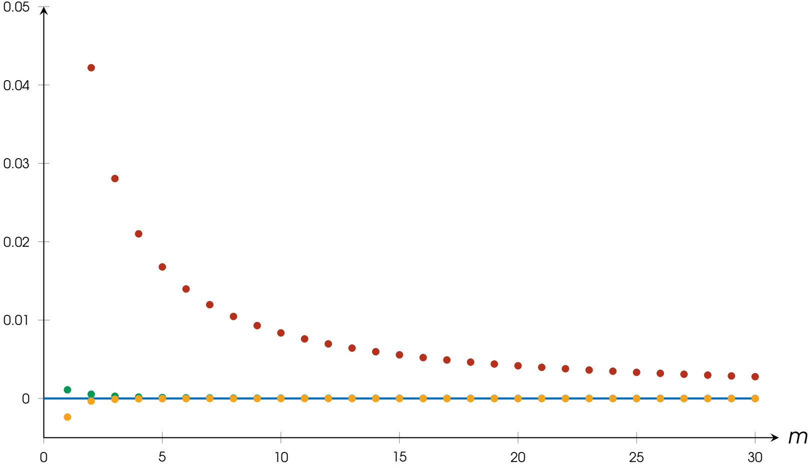
Visualising Stirling's formula

$$\frac{m!}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} - \left(1+\frac{1}{12}m^{-1}\right) = \mathrm{O}\left(m^{-2}\right)$$



Visualising Stirling's formula

$$\frac{m!}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} - \left(1 + \frac{1}{12}m^{-1} + \frac{1}{288}m^{-2}\right) = \mathrm{O}\left(m^{-3}\right)$$



ψ-class intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_1 + \cdots + d_n = 3g - 3 + n$$

- Volumes of moduli spaces of metric ribbon graphs
- Building block for all tautological intersection numbers:
  - Weil–Petersson volumes
  - Masur–Veech volumes
  - Hurwitz numbers
  - Gromov–Witten invariants for targets with s.s. quantum cohomology
  - ...
- Compute the perturbative expansion of topological 2d gravity

ψ-class intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_1 + \cdots + d_n = 3g - 3 + n$$

- Volumes of moduli spaces of metric ribbon graphs
- Building block for all tautological intersection numbers:
  - Weil–Petersson volumes
  - Masur–Veech volumes
  - Hurwitz numbers
  - Gromov–Witten invariants for targets with s.s. quantum cohomology
  - ...
- Compute the perturbative expansion of topological 2d gravity

ψ-class intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_1 + \cdots + d_n = 3g - 3 + n$$

- Volumes of moduli spaces of metric ribbon graphs
- Building block for all tautological intersection numbers:
  - Weil–Petersson volumes
  - Masur–Veech volumes
  - Hurwitz numbers
  - Gromov–Witten invariants for targets with s.s. quantum cohomology
  - ...
- Compute the perturbative expansion of topological 2d gravity

ψ-class intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_1 + \cdots + d_n = 3g - 3 + n$$

- Volumes of moduli spaces of metric ribbon graphs
- Building block for all tautological intersection numbers:
  - Weil–Petersson volumes
  - Masur–Veech volumes
  - Hurwitz numbers
  - Gromov–Witten invariants for targets with s.s. quantum cohomology
  - ...
- Compute the perturbative expansion of topological 2d gravity

Solution

Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i + 1)!!$

Witten conjecture/Kontsevich theorem, early '90s:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \sum_{m=2}^{n} (2d_m + 1) \langle\langle \tau_{d_1 + d_m - 1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle + \frac{1}{2} \sum_{a+b=d_1-2} \left(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\dots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle \langle\langle \tau_b \tau_{I_2} \rangle\rangle \right)$$

with initial data $\langle\langle \tau_0^3 \rangle\rangle = 1$ and $\langle\langle \tau_1 \rangle\rangle = \frac{1}{8}$.

Solution

Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i + 1)!!$

Witten conjecture/Kontsevich theorem, early '90s:

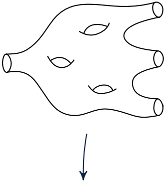
$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \sum_{m=2}^{n} (2d_m + 1) \langle\langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle + \frac{1}{2} \sum_{a+b=d_1-2} \Bigg(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\dots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle \langle\langle \tau_b \tau_{I_2} \rangle\rangle \Bigg)$$

with initial data $\langle\langle \tau_0^3 \rangle\rangle = 1$ and $\langle\langle \tau_1 \rangle\rangle = \frac{1}{8}$.

Solution

Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i + 1)!!$

Witten conjecture/Kontsevich theorem, early '90s:



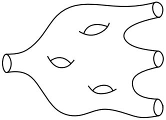
$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \sum_{m=2}^{n} (2d_m + 1) \langle\langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle + \frac{1}{2} \sum_{a+b=d_1-2} \left(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\ldots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle \langle\langle \tau_b \tau_{I_2} \rangle\rangle \right)$$

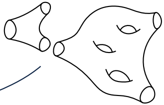
with initial data $\langle\langle \tau_0^3 \rangle\rangle = 1$ and $\langle\langle \tau_1 \rangle\rangle = \frac{1}{8}$.

Solution

Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i + 1)!!$

Witten conjecture/Kontsevich theorem, early ’90s:





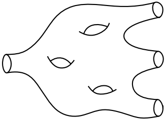
$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \sum_{m=2}^{n} (2d_m + 1) \langle\langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle + \frac{1}{2} \sum_{a+b=d_1-2} \left(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\ldots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle \langle\langle \tau_b \tau_{I_2} \rangle\rangle \right)$$

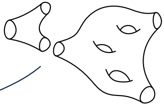
with initial data $\langle\langle \tau_0^3 \rangle\rangle = 1$ and $\langle\langle \tau_1 \rangle\rangle = \frac{1}{8}$.

Solution

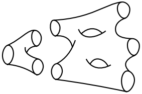
Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i + 1)!!$

Witten conjecture/Kontsevich theorem, early '90s:





$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \sum_{m=2}^{n} (2d_m + 1) \, \langle\langle \tau_{d_1 + d_m - 1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle + \frac{1}{2} \sum_{a+b=d_1-2} \left(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\dots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle \langle\langle \tau_b \tau_{I_2} \rangle\rangle \right)$$

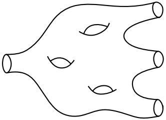


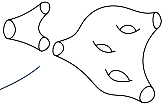
with initial data $\langle\langle \tau_0^3 \rangle\rangle = 1$ and $\langle\langle \tau_1 \rangle\rangle = \frac{1}{8}$.

Solution

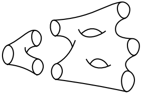
Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i + 1)!!$

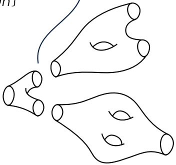
Witten conjecture/Kontsevich theorem, early '90s:





$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \sum_{m=2}^{n} (2d_m + 1) \, \langle\langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle$$
$$+ \frac{1}{2} \sum_{a+b=d_1-2} \left(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\dots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle \langle\langle \tau_b \tau_{I_2} \rangle\rangle \right)$$





with initial data $\langle\langle \tau_0^3 \rangle\rangle = 1$ and $\langle\langle \tau_1 \rangle\rangle = \frac{1}{8}$.

Large genus asymptotics

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\!\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\!\rangle = \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{(\frac{2}{3})^{2g-2+n}} \Big(1 + O(g^{-1})\Big)$$

- Conjectured by Delecroix–Goujard–Zograf–Zorich, 2019
- Proved by Aggarwal, 2020
  (combinatorial analysis of Witten–Kontsevich topological recursion)
- Re-proved by Guo–Yang, 2021
  (combinatorial analysis of the determinantal formula)

Questions

- Universal strategy, adaptable to different problems?
- 'Geometric' meaning of the formula?
- Subleading corrections?

Large genus asymptotics

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{\left(\frac{2}{3}\right)^{2g-2+n}} \left(1 + O(g^{-1})\right)$$

- Conjectured by Delecroix–Goujard–Zograf–Zorich, 2019
- Proved by Aggarwal, 2020
  (combinatorial analysis of Witten–Kontsevich topological recursion)
- Re-proved by Guo–Yang, 2021
  (combinatorial analysis of the determinantal formula)

Questions

- Universal strategy, adaptable to different problems?
- 'Geometric' meaning of the formula?
- Subleading corrections?

Large genus asymptotics

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{\left(\frac{2}{3}\right)^{2g-2+n}} \Big(1 + O(g^{-1})\Big)$$

- Conjectured by Delecroix–Goujard–Zograf–Zorich, 2019
- Proved by Aggarwal, 2020

  (combinatorial analysis of Witten–Kontsevich topological recursion)
- Re-proved by Guo–Yang, 2021

  (combinatorial analysis of the determinantal formula)

Questions

- Universal strategy, adaptable to different problems?
- ‘Geometric’ meaning of the formula?
- Subleading corrections?

Large genus asymptotics: new perspective

Answers

- Universal strategy: resurgent analysis of the determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \quad \xrightarrow{\text{quantisation}} \quad \left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$ (here $(x)^{\underline{k}} = x(x-1)\cdots(x-k+1)$):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n}\alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}}\alpha_k + O(g^{-k-1})\right)$$

Large genus asymptotics: new perspective

Answers

- Universal strategy: resurgent analysis of the determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \quad \xrightarrow{\text{quantisation}} \quad \left(\hbar^2 \frac{d^2}{dx^2} - x\right) \psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$ (here $(x)^{\underline{k}} = x(x-1)\cdots(x-k+1)$):

$$\langle\!\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\!\rangle = S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \, \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \, \alpha_k + O(g^{-k-1}) \right)$$

Large genus asymptotics: new perspective

Answers

- Universal strategy: resurgent analysis of the determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \xrightarrow{\text{quantisation}} \left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$ (here $(x)^{\underline{k}} = x(x-1)\cdots(x-k+1)$):

$$\langle\!\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\!\rangle = S \, \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n}\alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}}\alpha_k + O(g^{-k-1})\right)$$

$S = 1$
Stokes constant

Large genus asymptotics: new perspective

Answers

- Universal strategy: resurgent analysis of the determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \xrightarrow{\text{quantisation}} \left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$ (here $(x)^{\underline{k}} = x(x-1)\cdots(x-k+1)$):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S\,\frac{2^n}{4\pi}\,\frac{\Gamma(2g-2+n)}{A^{2g-2+n}}\left(1 + \frac{A}{2g-3+n}\,\alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}}\,\alpha_k + O(g^{-k-1})\right)$$

$A = 2/3$

$\psi \sim \frac{1}{\sqrt{2}x^{1/4}} e^{\pm \frac{A}{\hbar} x^{-3/2}}$

Large genus asymptotics: new perspective

Answers

- Universal strategy: resurgent analysis of the determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \quad \xrightarrow{\text{quantisation}} \quad \left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$ (here $(x)^{\underline{k}} = x(x-1)\cdots(x-k+1)$):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S\, \frac{2^n}{4\pi}\, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n}\alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}}\alpha_k + O(g^{-k-1})\right)$$

Computable; polynomial in n and multiplicities of d_i

Large genus asymptotics: new perspective

Answers

- Universal strategy: resurgent analysis of the determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \quad \xrightarrow{\text{quantisation}} \quad \left(\hbar^2 \frac{d^2}{dx^2} - x\right) \psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$ (here $(x)^{\underline{k}} = x(x-1)\cdots(x-k+1)$):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots \frac{A^k}{(2g-3+n)^{\underline{k}}} \alpha_k + O(g^{-k-1}) \right)$$

$$\alpha_1 = -\frac{17-15n+3n^2}{12} - \frac{(3-n)(n-p_0)}{2} - \frac{(n-p_0)^2}{4}$$

where $p_0 = \#\{ d_i = 0 \}$

Visualising the large genus asymptotics

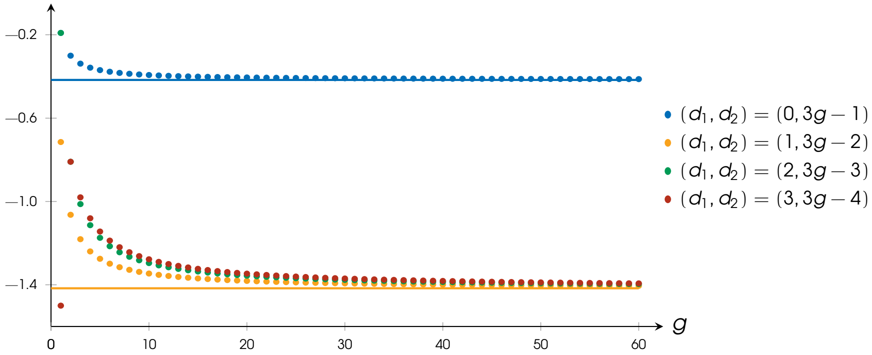
$$\frac{2g-3+n}{2/3}\left(\frac{\langle\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\rangle}{\frac{2^n}{4\pi}\frac{\Gamma(2g-2+n)}{(2/3)^{2g-2+n}}}-1\right)=\alpha_1(n,p_0)+O(g^{-1})$$

For $n=2$:

Visualising the large genus asymptotics

$$\frac{2g-3+n}{2/3}\left(\frac{\langle\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\rangle}{\frac{2^n}{4\pi}\frac{\Gamma(2g-2+n)}{(2/3)^{2g-2+n}}}-1\right)=\alpha_1(n,p_0)+O(g^{-1})$$

For $n=2$:



Darboux meets Borel

Darboux's idea:

- Convergent power series:

$$\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$$

- Get a holomorphic function around the origin, take analytic continuation
- The large m asymptotics of a_m is totally controlled by the behaviour of $\widehat{\varphi}$ at its singularities

Borel's idea:

- Divergent power series:

$$\widetilde{\varphi}(\hbar) = \sum_{m \geqslant 0} a_m \hbar^m, \qquad a_m = O(A^{-m} m!)$$

- The Borel transform

$$\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$$

is now convergent

- Apply Darboux's idea

Darboux meets Borel

Darboux's idea:

- Convergent power series:

$$\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$$

- Get a holomorphic function around the origin, take analytic continuation
- The large m asymptotics of a_m is totally controlled by the behaviour of $\widehat{\varphi}$ at its singularities

Borel's idea:

- Divergent power series:

$$\widetilde{\varphi}(\hbar) = \sum_{m \geqslant 0} a_m \hbar^m, \quad a_m = O(A^{-m} m!)$$

- The Borel transform

$$\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$$

  is now convergent
- Apply Darboux's idea

Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = (\text{holomorphic } @A) \log(s - A) + \text{holomorphic } @A$$

$$a_m = \frac{m!}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$

Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m\geqslant 0} \frac{a_m}{m!} s^m$

Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = (\text{holomorphic @}A) \log(s-A) + \text{holomorphic @}A$$

$$a_m = \frac{m!}{2\pi i} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$

Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi}\,\widehat{\psi}(s-A)\log(s-A) + \text{holomorphic } @A$$

$$a_m = \frac{m!}{2\pi i}\oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}}\,ds$$

Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi}\,\widehat{\psi}(s - A)\log(s - A) + \text{holomorphic } @A$$

$$a_m = \frac{m!}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}}\, ds$$

Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi} \widehat{\psi}(s - A) \log(s - A) + \text{holomorphic } @A$$

Stokes constant
$S \in \mathbb{C}$

$$a_m = \frac{m!}{2\pi i} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$

Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

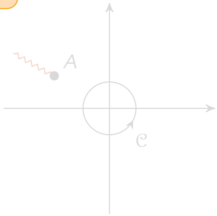
Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi} \widehat{\psi}(s - A) \log(s - A) + \text{holomorphic } @A$$

Stokes constant
$S \in \mathbb{C}$

minor
holom @origin
$\widehat{\psi}(s) = \sum_{k \geqslant 0} \frac{b_k}{k!} s^k$

$$a_m = \frac{m!}{2\pi i} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$



Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

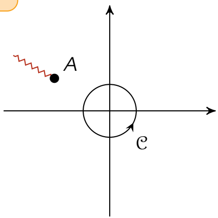
Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi}\, \widehat{\psi}(s-A) \log(s-A) + \text{holomorphic } @A$$

Stokes constant $S \in \mathbb{C}$

minor
holom @origin
$\widehat{\psi}(s) = \sum_{k \geqslant 0} \frac{b_k}{k!} s^k$

$$a_m = \frac{m!}{2\pi \mathrm{i}} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$



Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

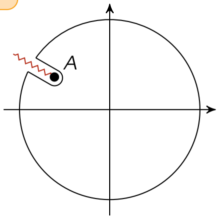
Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi} \widehat{\psi}(s - A) \log(s - A) + \text{holomorphic @}A$$

Stokes constant $S \in \mathbb{C}$

minor
holom @origin
$\widehat{\psi}(s) = \sum_{k \geqslant 0} \frac{b_k}{k!} s^k$

$$a_m = \frac{m!}{2\pi i} \oint_{\mathcal{C}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$



Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

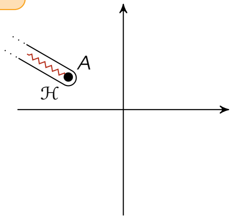
Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi} \widehat{\psi}(s - A) \log(s - A) + \text{holomorphic } @A$$

Stokes constant $S \in \mathbb{C}$

minor
holom @origin
$\widehat{\psi}(s) = \sum_{k \geqslant 0} \frac{b_k}{k!} s^k$

$$a_m = \frac{m!}{2\pi \mathrm{i}} \int_{\mathcal{H}} \frac{\widehat{\varphi}(s)}{s^{m+1}} ds$$



Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

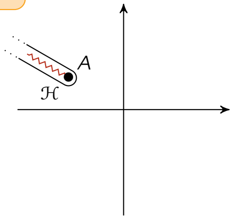
Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi} \widehat{\psi}(s - A) \log(s - A) + \text{holomorphic } @A$$

Stokes constant
$S \in \mathbb{C}$

minor
holom @origin
$\widehat{\psi}(s) = \sum_{k \geqslant 0} \frac{b_k}{k!} s^k$

$$a_m = -\frac{m!}{2\pi \mathrm{i}} \int_{\mathcal{H}} \frac{S}{2\pi} \frac{\widehat{\psi}(s - A)}{s^{m+1}} \log(s - A) ds$$



Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m\geqslant 0} \frac{a_m}{m!} s^m$

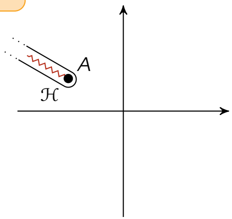
Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi}\,\widehat{\psi}(s-A)\log(s-A) + \text{holomorphic } @A$$

Stokes constant
$S \in \mathbb{C}$

minor
holom @origin
$\widehat{\psi}(s) = \sum_{k\geqslant 0} \frac{b_k}{k!} s^k$

$$a_m = \frac{S}{2\pi} m! \int_A^\infty \frac{\widehat{\psi}(s-A)}{s^{m+1}} ds$$



Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

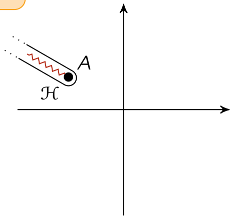
Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = -\frac{S}{2\pi}\,\widehat{\psi}(s-A)\log(s-A) + \text{holomorphic @}A$$

Stokes constant $S \in \mathbb{C}$

minor
holom @origin
$\widehat{\psi}(s) = \sum_{k \geqslant 0} \frac{b_k}{k!} s^k$

$$a_m = \frac{S}{2\pi}\frac{\Gamma(m)}{A^m}\Big(b_0 + \frac{A}{m-1}b_1 + \cdots + \frac{A^k}{(m-1)^{\underline{k}}}b_k + \mathrm{O}(m^{-k-1})\Big)$$



Darboux meets Borel: summary

- Given: $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m$ divergent
- Borel transform: $\widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$ convergent
- Suppose you can compute:
  1. Log singularities: $A_1, \ldots, A_n$
  2. Stokes constants: $S_1, \ldots, S_n$
  3. Minors: $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$
- Large m asymptotics:

$$a_m = \frac{\Gamma(m)}{2\pi}\left(\frac{S_1}{A_1^m}\left(b_{1,0} + \frac{A_1}{m-1}b_{1,1} + \frac{A_1^2}{(m-1)(m-2)}b_{1,2} + \cdots\right) + \cdots + \frac{S_n}{A_n^m}\left(b_{n,0} + \frac{A_n}{m-1}b_{n,1} + \frac{A_n^2}{(m-1)(m-2)}b_{n,2} + \cdots\right)\right)$$

Darboux meets Borel: summary

- Given: $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m$ divergent
- Borel transform: $\widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$ convergent
- Suppose you can compute:
  1. Log singularities: $A_1, \ldots, A_n$
  2. Stokes constants: $S_1, \ldots, S_n$
  3. Minors: $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$
- Large m asymptotics:

$$a_m = \frac{\Gamma(m)}{2\pi}\left(\frac{S_1}{A_1^m}\left(b_{1,0} + \frac{A_1}{m-1} b_{1,1} + \frac{A_1^2}{(m-1)(m-2)} b_{1,2} + \cdots\right) + \cdots + \frac{S_n}{A_n^m}\left(b_{n,0} + \frac{A_n}{m-1} b_{n,1} + \frac{A_n^2}{(m-1)(m-2)} b_{n,2} + \cdots\right)\right)$$

Darboux meets Borel: summary

- Given: $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m$ divergent
- Borel transform: $\widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$ convergent
- Suppose you can compute:
  1. Log singularities: $A_1, \ldots, A_n$
  2. Stokes constants: $S_1, \ldots, S_n$
  3. Minors: $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$
- Large m asymptotics:

$$a_m = \frac{\Gamma(m)}{2\pi}\left(\frac{S_1}{A_1^m}\left(b_{1,0} + \frac{A_1}{m-1} b_{1,1} + \frac{A_1^2}{(m-1)(m-2)} b_{1,2} + \cdots\right) + \cdots + \frac{S_n}{A_n^m}\left(b_{n,0} + \frac{A_n}{m-1} b_{n,1} + \frac{A_n^2}{(m-1)(m-2)} b_{n,2} + \cdots\right)\right)$$

Darboux meets Borel: summary

- Given: $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m$ divergent
- Borel transform: $\widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$ convergent
- Suppose you can compute:
  1. Log singularities: $A_1, \ldots, A_n$
  2. Stokes constants: $S_1, \ldots, S_n$
  3. Minors: $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$
- Large m asymptotics:

$$
\begin{aligned}
a_m = \frac{\Gamma(m)}{2\pi} \Bigg(& \frac{S_1}{A_1^m} \left(b_{1,0} + \frac{A_1}{m-1} b_{1,1} + \frac{A_1^2}{(m-1)(m-2)} b_{1,2} + \cdots \right) \\
& + \cdots \\
& + \frac{S_n}{A_n^m} \left(b_{n,0} + \frac{A_n}{m-1} b_{n,1} + \frac{A_n^2}{(m-1)(m-2)} b_{n,2} + \cdots \right) \Bigg)
\end{aligned}
$$

Main example: Airy function

The formal (WKB) solutions of the Airy ODE, $\left(\hbar^2 \frac{d^2}{dx^2} - x\right) \psi(x, \hbar) = 0$, are

$$\psi_{\text{Ai}}(x;\hbar) = \frac{e^{-\frac{2}{3\hbar}x^{3/2}}}{\sqrt{2}x^{1/4}} \left(1 - \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right)$$

$$\psi_{\text{Bi}}(x;\hbar) = \psi_{\text{Ai}}(x;-\hbar)$$

$\widetilde{\psi}_{\text{Ai}}$ is a divergent series in $\hbar$. Its Borel transform has a single log singularity at $s = +\frac{4}{3}x^{3/2}$:

$$\widehat{\psi}_{\text{Ai}}(x;s) = -\frac{1}{2\pi}\widehat{\psi}_{\text{Bi}}(s - \tfrac{4}{3}x^{3/2}) \log(s - \tfrac{4}{3}x^{3/2}) + \text{holom } @ s = +\tfrac{4}{3}x^{3/2}$$

For the Airy function: (sing, Stokes cnstn, minor) $= (+\frac{4}{3}x^{3/2}, 1, \widehat{\psi}_{\text{Bi}})$

Main example: Airy function

The formal (WKB) solutions of the Airy ODE, $\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$, are

$$\psi_{\mathrm{Ai}}(x;\hbar) = \frac{e^{-\frac{2}{3\hbar}x^{3/2}}}{\sqrt{2}x^{1/4}}\left(1 - \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right)$$

$$\psi_{\mathrm{Bi}}(x;\hbar) = \psi_{\mathrm{Ai}}(x;-\hbar)$$

$\widetilde{\psi}_{\mathrm{Ai}}$ is a divergent series in $\hbar$. Its Borel transform has a single log singularity at $s = +\frac{4}{3}x^{3/2}$:

$$\widehat{\psi}_{\mathrm{Ai}}(x;s) = -\frac{1}{2\pi}\widehat{\psi}_{\mathrm{Bi}}(s - \tfrac{4}{3}x^{3/2})\log(s - \tfrac{4}{3}x^{3/2}) + \text{holom } @ s = +\tfrac{4}{3}x^{3/2}$$

For the Airy function: (sing, Stokes cnstn, minor) $= (+\frac{4}{3}x^{3/2}, 1, \widehat{\psi}_{\mathrm{Bi}})$

Main example: Airy function

The formal (WKB) solutions of the Airy ODE, $\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$, are

$$\psi_{\mathrm{Ai}}(x;\hbar) = \frac{e^{-\frac{2}{3\hbar}x^{3/2}}}{\sqrt{2}x^{1/4}}\left(1 - \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right) = e^{-\frac{2}{3\hbar}x^{3/2}}\,\widetilde{\psi}_{\mathrm{Ai}}(x;\hbar)$$

$$\psi_{\mathrm{Bi}}(x;\hbar) = \psi_{\mathrm{Ai}}(x;-\hbar)$$

$\widetilde{\psi}_{\mathrm{Ai}}$ is a divergent series in $\hbar$. Its Borel transform has a single log singularity at $s = +\frac{4}{3}x^{3/2}$:

$$\widehat{\psi}_{\mathrm{Ai}}(x;s) = -\frac{1}{2\pi}\,\widehat{\psi}_{\mathrm{Bi}}(s - \tfrac{4}{3}x^{3/2})\log(s - \tfrac{4}{3}x^{3/2}) + \text{holom } @ s = +\tfrac{4}{3}x^{3/2}$$

For the Airy function: (sing, Stokes cnstn, minor) $= (+\frac{4}{3}x^{3/2}, 1, \widehat{\psi}_{\mathrm{Bi}})$

Main example: Airy function

The formal (WKB) solutions of the Airy ODE, $\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$, are

$$\psi_{\mathrm{Ai}}(x;\hbar) = \frac{e^{-\frac{2}{3\hbar}x^{3/2}}}{\sqrt{2}x^{1/4}}\left(1 - \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right) = e^{-\frac{2}{3\hbar}x^{3/2}}\,\widetilde{\psi}_{\mathrm{Ai}}(x;\hbar)$$

$$\psi_{\mathrm{Bi}}(x;\hbar) = \psi_{\mathrm{Ai}}(x;-\hbar)$$

$\widetilde{\psi}_{\mathrm{Ai}}$ is a divergent series in $\hbar$. Its Borel transform has a single log singularity at $s = +\frac{4}{3}x^{3/2}$:

$$\widehat{\psi}_{\mathrm{Ai}}(x;s) = -\frac{1}{2\pi}\,\widehat{\psi}_{\mathrm{Bi}}(s - \tfrac{4}{3}x^{3/2})\log(s - \tfrac{4}{3}x^{3/2}) + \text{holom } @ s = +\tfrac{4}{3}x^{3/2}$$

For the Airy function: (sing, Stokes cnstn, minor) $= (+\frac{4}{3}x^{3/2}, 1, \widehat{\psi}_{\mathrm{Bi}})$

Main example: Airy function

The formal (WKB) solutions of the Airy ODE, $\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$, are

$$\psi_{\mathrm{Ai}}(x;\hbar) = \frac{e^{-\frac{2}{3\hbar}x^{3/2}}}{\sqrt{2}x^{1/4}}\left(1 - \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right) = e^{-\frac{2}{3\hbar}x^{3/2}}\,\widetilde{\psi}_{\mathrm{Ai}}(x;\hbar)$$

$$\psi_{\mathrm{Bi}}(x;\hbar) = \psi_{\mathrm{Ai}}(x;-\hbar)$$

$\widetilde{\psi}_{\mathrm{Ai}}$ is a divergent series in $\hbar$. Its Borel transform has a single log singularity at $s = +\frac{4}{3}x^{3/2}$:

$$\widehat{\psi}_{\mathrm{Ai}}(x;s) = -\frac{1}{2\pi}\,\widehat{\psi}_{\mathrm{Bi}}\left(s - \tfrac{4}{3}x^{3/2}\right)\log\left(s - \tfrac{4}{3}x^{3/2}\right) + \text{holom } @s = +\tfrac{4}{3}x^{3/2}$$

Bairy fnct resurges at the sing of Airy fnct

For the Airy function: (sing, Stokes cnstn, minor) $= \left(+\frac{4}{3}x^{3/2}, 1, \widehat{\psi}_{\mathrm{Bi}}\right)$

Main example: Airy function

The formal (WKB) solutions of the Airy ODE, $\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$, are

$$\psi_{\mathrm{Ai}}(x;\hbar) = \frac{e^{-\frac{2}{3\hbar}x^{3/2}}}{\sqrt{2}x^{1/4}}\left(1 - \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right) = e^{-\frac{2}{3\hbar}x^{3/2}}\,\widetilde{\psi}_{\mathrm{Ai}}(x;\hbar)$$

$$\psi_{\mathrm{Bi}}(x;\hbar) = \psi_{\mathrm{Ai}}(x;-\hbar)$$

$\widetilde{\psi}_{\mathrm{Ai}}$ is a divergent series in $\hbar$. Its Borel transform has a single log singularity at $s = +\frac{4}{3}x^{3/2}$:

$$\widehat{\psi}_{\mathrm{Ai}}(x;s) = -\frac{1}{2\pi}\,\widehat{\psi}_{\mathrm{Bi}}\left(s - \tfrac{4}{3}x^{3/2}\right)\log\left(s - \tfrac{4}{3}x^{3/2}\right) + \text{holom } @s = +\tfrac{4}{3}x^{3/2}$$

Bairy fnct resurges
at the sing of Airy fnct

For the Airy function: (sing, Stokes cnstn, minor) $= \left(+\frac{4}{3}x^{3/2}, 1, \widehat{\psi}_{\mathrm{Bi}}\right)$

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} W_{g,n}(x_1, \ldots, x_n)$$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure

3. Get the large genus asymptotics (with subleading contributions!)

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} W_{g,n}(x_1, \ldots, x_n)$$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure
3. Get the large genus asymptotics (with subleading contributions!)

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} \, W_{g,n}(x_1, \ldots, x_n)$$

$$= (-2)^{-(2g-2+n)} \sum_{d_1, \ldots, d_n} \frac{\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle}{2x_1^{d_1+3/2} \cdots 2x_n^{d_n+3/2}}$$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure
3. Get the large genus asymptotics (with subleading contributions!)

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} W_{g,n}(x_1, \ldots, x_n)$$

$$= (-2)^{-(2g-2+n)} \sum_{d_1,\ldots,d_n} \frac{\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle}{2x_1^{d_1+3/2} \cdots 2x_n^{d_n+3/2}}$$

n fixed
$\hbar \to$ genus
$x_i \to d_i$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure

3. Get the large genus asymptotics (with subleading contributions!)

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} W_{g,n}(x_1, \ldots, x_n)$$

$$= (-2)^{-(2g-2+n)} \sum_{d_1,\ldots,d_n} \frac{\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle}{2x_1^{d_1+3/2} \cdots 2x_n^{d_n+3/2}}$$

n fixed
$\hbar \to$ genus
$x_i \to d_i$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure

3. Get the large genus asymptotics (with subleading contributions!)

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} \, W_{g,n}(x_1, \ldots, x_n)$$

$$= (-2)^{-(2g-2+n)} \sum_{d_1, \ldots, d_n} \frac{\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle}{2x_1^{d_1+3/2} \cdots 2x_n^{d_n+3/2}}$$

n fixed
$\hbar \to$ genus
$x_i \to d_i$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure
3. Get the large genus asymptotics (with subleading contributions!)

Determinantal formula: setup

Arrange the formal Airy functions as

$$\Psi(x,\hbar) = \begin{pmatrix} \psi_{\mathrm{Ai}} & \psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}} & \psi'_{\mathrm{Bi}} \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C})\,.$$

It solves the system $\left(\hbar\frac{d}{dx} - \left(\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right)\right)\Psi = 0$.

Define the matrix

$$M(x,\hbar) = \Psi(x,\hbar)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\Psi^{-1}(x,\hbar)$$
$$= \begin{pmatrix} \frac{1}{2}(\psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}} + \psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}}) & \psi_{\mathrm{Ai}}\psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} & -\frac{1}{2}(\psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} + \psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}}) \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})\,.$$

Crucial facts:

- M contains only quadratic products Ai-Bi
- The exponential terms cancel out $\implies$ M is a formal series in $\hbar$.

Determinantal formula: setup

Arrange the formal Airy functions as

$$\Psi(x,\hbar) = \begin{pmatrix} \psi_{\mathrm{Ai}} & \psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}} & \psi'_{\mathrm{Bi}} \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C})\,.$$

It solves the system $\left(\hbar\frac{d}{dx} - \left(\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right)\right)\Psi = 0$.

Define the matrix

$$\begin{aligned} M(x,\hbar) &= \Psi(x,\hbar) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \Psi^{-1}(x,\hbar) \\ &= \begin{pmatrix} \frac{1}{2}(\psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}} + \psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}}) & \psi_{\mathrm{Ai}}\psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} & -\frac{1}{2}(\psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} + \psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}}) \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})\,. \end{aligned}$$

Crucial facts:

- M contains only quadratic products Ai-Bi
- The exponential terms cancel out $\implies$ M is a formal series in $\hbar$.

Determinantal formula: setup

Arrange the formal Airy functions as

$$\Psi(x,\hbar) = \begin{pmatrix} \psi_{\mathrm{Ai}} & \psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}} & \psi'_{\mathrm{Bi}} \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C})\,.$$

It solves the system $\left(\hbar \frac{d}{dx} - \left(\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right)\right)\Psi = 0$.

Define the matrix

$$\begin{aligned} M(x,\hbar) &= \Psi(x,\hbar) \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix} \Psi^{-1}(x,\hbar) \\ &= \begin{pmatrix} \frac{1}{2}(\psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}} + \psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}}) & \psi_{\mathrm{Ai}}\psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} & -\frac{1}{2}(\psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} + \psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}}) \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})\,. \end{aligned}$$

Crucial facts:

- M contains only quadratic products Ai-Bi
- The exponential terms cancel out $\implies$ M is a formal series in $\hbar$.

Determinantal formula: setup

Arrange the formal Airy functions as

$$\Psi(x,\hbar) = \begin{pmatrix} \psi_{\mathrm{Ai}} & \psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}} & \psi'_{\mathrm{Bi}} \end{pmatrix} \in \mathbf{SL}(2,\mathbb{C})\,.$$

It solves the system $\left(\hbar\frac{d}{dx} - \left(\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right)\right)\Psi = 0$.

Define the matrix

$$\begin{aligned} M(x,\hbar) &= \Psi(x,\hbar)\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}\Psi^{-1}(x,\hbar) \\ &= \begin{pmatrix} \frac{1}{2}(\widetilde{\psi}'_{\mathrm{Ai}}\widetilde{\psi}_{\mathrm{Bi}} + \widetilde{\psi}_{\mathrm{Ai}}\widetilde{\psi}'_{\mathrm{Bi}}) & \widetilde{\psi}_{\mathrm{Ai}}\widetilde{\psi}_{\mathrm{Bi}} \\ \widetilde{\psi}'_{\mathrm{Ai}}\widetilde{\psi}'_{\mathrm{Bi}} & -\frac{1}{2}(\widetilde{\psi}_{\mathrm{Ai}}\widetilde{\psi}'_{\mathrm{Bi}} + \widetilde{\psi}'_{\mathrm{Ai}}\widetilde{\psi}_{\mathrm{Bi}}) \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C})\,. \end{aligned}$$

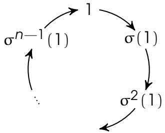
Crucial facts:

- M contains only quadratic products Ai-Bi
- The exponential terms cancel out $\Longrightarrow$ M is a formal series in $\hbar$.

Determinantal formula: result

Cyclic permutations:

$C_n = \{ \sigma \in S_n \mid \text{no fxd prpr sbsts} \}$



Determinantal formula (Bergère–Eynard, Bertola–Dubrovin–Yang):

$$W_n(x_1, \ldots, x_n; \hbar) = (-1)^{n-1} \sum_{\sigma \in C_n} \frac{\mathrm{Tr}\big(M(x_1,\hbar)M(x_{\sigma(1)},\hbar)\cdots M(x_{\sigma^{n-1}(1)},\hbar)\big)}{(x_1 - x_{\sigma(1)})(x_2 - x_{\sigma(2)})\cdots(x_n - x_{\sigma(n)})}$$

Example: $n = 2$

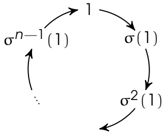
$$W_2 = -\frac{\mathrm{Tr}\big(M(x_1,\hbar)M(x_2,\hbar)\big)}{(x_1 - x_2)(x_2 - x_1)}$$

$$= \frac{\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}_{\mathrm{Bi},1}\widetilde{\psi}'_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Bi},2}\widetilde{\psi}'_{\mathrm{Ai},2}}{(x_1 - x_2)^2} + (x_1 \leftrightarrow x_2)$$

Determinantal formula: result

Cyclic permutations:

$C_n = \{\, \sigma \in S_n \mid \text{no fxd prpr sbsts} \,\}$



Determinantal formula (Bergère–Eynard, Bertola–Dubrovin–Yang):

$$W_n(x_1, \dots, x_n; \hbar) = (-1)^{n-1} \sum_{\sigma \in C_n} \frac{\mathrm{Tr}\big(M(x_1,\hbar)M(x_{\sigma(1)},\hbar)\cdots M(x_{\sigma^{n-1}(1)},\hbar)\big)}{(x_1 - x_{\sigma(1)})(x_2 - x_{\sigma(2)})\cdots(x_n - x_{\sigma(n)})}$$

Example: $n = 2$

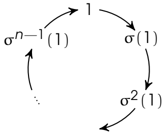
$$W_2 = -\frac{\mathrm{Tr}\big(M(x_1,\hbar)M(x_2,\hbar)\big)}{(x_1 - x_2)(x_2 - x_1)}$$

$$= \frac{\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}_{\mathrm{Bi},1}\widetilde{\psi}'_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Bi},2}\widetilde{\psi}'_{\mathrm{Ai},2}}{(x_1 - x_2)^2} + (x_1 \leftrightarrow x_2)$$

Determinantal formula: result

Cyclic permutations:

$C_n = \{ \sigma \in S_n \mid \text{no fxd prpr sbsts} \}$



Determinantal formula (Bergère–Eynard, Bertola–Dubrovin–Yang):

$$W_n(x_1, \dots, x_n; \hbar) = (-1)^{n-1} \sum_{\sigma \in C_n} \frac{\operatorname{Tr}\big(M(x_1,\hbar)M(x_{\sigma(1)},\hbar)\cdots M(x_{\sigma^{n-1}(1)},\hbar)\big)}{(x_1 - x_{\sigma(1)})(x_2 - x_{\sigma(2)})\cdots(x_n - x_{\sigma(n)})}$$

Example: $n = 2$

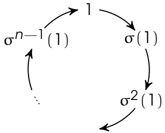
$$W_2 = -\frac{\operatorname{Tr}\big(M(x_1,\hbar)M(x_2,\hbar)\big)}{(x_1 - x_2)(x_2 - x_1)}$$

$$= \frac{\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}_{\mathrm{Bi},1}\widetilde{\psi}'_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Bi},2}\widetilde{\psi}'_{\mathrm{Ai},2}}{(x_1 - x_2)^2} + (x_1 \leftrightarrow x_2)$$

Determinantal formula: result

Cyclic permutations:

$C_n = \{\, \sigma \in S_n \mid \text{no fxd prpr sbsts} \,\}$



Determinantal formula (Bergère–Eynard, Bertola–Dubrovin–Yang):

$$W_n(x_1,\dots,x_n;\hbar) = (-1)^{n-1} \sum_{\sigma\in C_n} \frac{\mathrm{Tr}\big(M(x_1,\hbar)M(x_{\sigma(1)},\hbar)\cdots M(x_{\sigma^{n-1}(1)},\hbar)\big)}{(x_1-x_{\sigma(1)})(x_2-x_{\sigma(2)})\cdots(x_n-x_{\sigma(n)})}$$

Example: $n=2$

$$W_2 = -\frac{\mathrm{Tr}\big(M(x_1,\hbar)M(x_2,\hbar)\big)}{(x_1-x_2)(x_2-x_1)}$$

$$= \frac{\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}_{\mathrm{Bi},1}\widetilde{\psi}'_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Ai},2}\widetilde{\psi}'_{\mathrm{Bi},2} + \frac{1}{2}\widetilde{\psi}_{\mathrm{Ai},1}\widetilde{\psi}'_{\mathrm{Bi},1}\widetilde{\psi}_{\mathrm{Bi},2}\widetilde{\psi}'_{\mathrm{Ai},2}}{(x_1-x_2)^2} + (x_1 \leftrightarrow x_2)$$

Singularity structure of $\widehat{W}_n$

Singularity strct of $\widehat{\psi}_{\mathrm{Ai}}, \widehat{\psi}_{\mathrm{Bi}}$ $\Longrightarrow$ Singularity strct of $\widehat{W}_n$

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\tfrac{4}{3}x_i^{3/2} \quad \text{and} \quad -\tfrac{4}{3}x_i^{3/2}, \qquad i = 1, \ldots, n$$

- Stokes constants: $S = 1$
- Minors:
  - (A) at $+\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$
  - (B) at $-\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$

Singularity structure of $\widehat{W}_n$

Singularity strct of $\widehat{\psi}_{\mathrm{Ai}}$, $\widehat{\psi}_{\mathrm{Bi}}$ $\Longrightarrow$ Singularity strct of $\widehat{W}_n$

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\tfrac{4}{3}x_i^{3/2} \quad \text{and} \quad -\tfrac{4}{3}x_i^{3/2}, \qquad i = 1, \ldots, n$$

- Stokes constants: $S = 1$
- Minors:
  - (A) at $+\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$
  - (B) at $-\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$

Singularity structure of $\widehat{W}_n$

Singularity strct of $\widehat{\psi}_{\mathrm{Ai}}$, $\widehat{\psi}_{\mathrm{Bi}}$ $\Longrightarrow$ Singularity strct of $\widehat{W}_n$

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\tfrac{4}{3}x_i^{3/2} \quad \text{and} \quad -\tfrac{4}{3}x_i^{3/2}, \qquad i = 1, \ldots, n$$

- Stokes constants: $S = 1$
- Minors:
  - (A) at $+\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$
  - (B) at $-\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$

Singularity structure of $\widehat{W}_n$

Singularity strct of $\widehat{\psi}_{\mathrm{Ai}}, \widehat{\psi}_{\mathrm{Bi}}$ $\Longrightarrow$ Singularity strct of $\widehat{W}_n$

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\tfrac{4}{3}x_i^{3/2} \quad \text{and} \quad -\tfrac{4}{3}x_i^{3/2}, \qquad i = 1, \ldots, n$$

- Stokes constants: $S = 1$
- Minors:
  - A at $+\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$
  - B at $-\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$

Summary

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \, \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \, \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 1$

  Stokes constants of the Airy ODE
- $A = 2/3$

  leading exp behaviour of Ai
- α_k polynomials in n and multiplicities of d_i (conj by Guo–Yang)

  are computable from the asymptotic expansion coeffs of Ai

Summary

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \, \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \, \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 1$

  Stokes constants of the Airy ODE
- $A = 2/3$

  leading exp behaviour of Ai
- α_k polynomials in n and multiplicities of d_i (conj by Guo–Yang)

  are computable from the asymptotic expansion coeffs of Ai

Summary

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \, \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \, \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 1$

  Stokes constants of the Airy ODE
- $A = 2/3$

  leading exp behaviour of Ai
- α_k polynomials in n and multiplicities of d_i (conj by Guo–Yang)

  are computable from the asymptotic expansion coeffs of Ai

Summary

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \, \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)_{\underline{k}}} \, \alpha_k + \mathrm{O}(g^{-k-1}) \right)$$

where:

- $S = 1$

  Stokes constants of the Airy ODE

- $A = 2/3$

  leading exp behaviour of Ai

- α_k polynomials in n and multiplicities of d_i (conj by Guo–Yang)

  are computable from the asymptotic expansion coeffs of Ai

Bessel

Norbury's intersection numbers (super WP volumes, BGW tau function):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle^{\Theta} = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i + 1)!!$$

$$= S \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 2$

  Stokes constants of the Bessel ODE
- $A = 2$

  leading exp behaviour of K_0
- α_k polynomials in n and multiplicities of d_i

  are computable from the asymptotic expansion coeffs of K_0

Bessel

Norbury's intersection numbers (super WP volumes, BGW tau function):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle^{\Theta} = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i+1)!!$$

$$= S \, \frac{2^n}{4\pi} \, \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 2$

  Stokes constants of the Bessel ODE
- $A = 2$

  leading exp behaviour of K_0
- α_k polynomials in n and multiplicities of d_i

  are computable from the asymptotic expansion coeffs of K_0

Bessel

Norbury's intersection numbers (super WP volumes, BGW tau function):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle^{\Theta} = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i+1)!!$$

$$= S \, \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n} \alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}} \alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 2$

  Stokes constants of the Bessel ODE
- $A = 2$

  leading exp behaviour of K_0
- α_k polynomials in n and multiplicities of d_i

  are computable from the asymptotic expansion coeffs of K_0

r-Airy

Witten's r-spin intersection numbers:

$$\langle\langle \tau_{a_1,d_1}\cdots\tau_{a_n,d_n}\rangle\rangle^{r\text{-spin}} = \int_{\overline{\mathcal{M}}_{g,n}} c_W(a_1,\ldots,a_n)\prod_{i=1}^{n}\psi_i^{d_i}(rd_i+a_i)!_{(r)}$$

$$= \frac{2^n}{2\pi}\frac{\Gamma(2g-2+n)}{r^{g-1-|d|}}\Bigg[\frac{S_{r,1}}{|A_{r,1}|^{2g-2+n}}\left(\alpha_0^{(r,1)}+\frac{|A_{r,1}|}{2g-3+n)}\alpha_1^{(r,1)}+\cdots\right)$$

$$+\cdots$$

$$+\frac{S_{r,\lfloor\frac{r-1}{2}\rfloor}}{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^{2g-2+n}}\left(\alpha_0^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\frac{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^K}{2g-3+n}\alpha_1^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\cdots\right)$$

$$+\frac{\delta_r^{\text{even}}}{2}\frac{S_{r,\frac{r}{2}}}{|A_{r,\frac{r}{2}}|^{2g-2+n}}\left(\alpha_0^{(r,\frac{r}{2})}+\frac{|A_{r,\frac{r}{2}}|^K}{2g-3+n}\alpha_1^{(r,\frac{r}{2})}+\cdots\right)\Bigg]$$

where $S_{r,\alpha}$, $A_{r,\alpha}$, $\alpha_k^{(r,\alpha)}$ are obtained the r-Airy ODE.

r-Airy

Witten's r-spin intersection numbers:

$$\langle\langle \tau_{a_1,d_1}\cdots\tau_{a_n,d_n}\rangle\rangle^{r\text{-spin}} = \int_{\overline{\mathcal{M}}_{g,n}} c_W(a_1,\ldots,a_n)\prod_{i=1}^{n}\psi_i^{d_i}(rd_i+a_i)!_{(r)}$$

$$= \frac{2^n}{2\pi}\frac{\Gamma(2g-2+n)}{r^{g-1-|d|}}\Bigg[\frac{S_{r,1}}{|A_{r,1}|^{2g-2+n}}\left(\alpha_0^{(r,1)}+\frac{|A_{r,1}|}{2g-3+n)}\alpha_1^{(r,1)}+\cdots\right)$$

$$+\cdots$$

$$+\frac{S_{r,\lfloor\frac{r-1}{2}\rfloor}}{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^{2g-2+n}}\left(\alpha_0^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\frac{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^K}{2g-3+n}\alpha_1^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\cdots\right)$$

$$+\frac{\delta_r^{\text{even}}}{2}\frac{S_{r,\frac{r}{2}}}{|A_{r,\frac{r}{2}}|^{2g-2+n}}\left(\alpha_0^{(r,\frac{r}{2})}+\frac{|A_{r,\frac{r}{2}}|^K}{2g-3+n}\alpha_1^{(r,\frac{r}{2})}+\cdots\right)\Bigg]$$

where $S_{r,\alpha}$, $A_{r,\alpha}$, $\alpha_k^{(r,\alpha)}$ are obtained the r-Airy ODE.

r-Airy

Witten's r-spin intersection numbers:

$$\langle\langle \tau_{a_1,d_1}\cdots\tau_{a_n,d_n}\rangle\rangle^{r\text{-spin}} = \int_{\overline{\mathcal{M}}_{g,n}} c_W(a_1,\dots,a_n)\prod_{i=1}^{n}\psi_i^{d_i}(rd_i+a_i)!_{(r)}$$

$$= \frac{2^n}{2\pi}\frac{\Gamma(2g-2+n)}{r^{g-1-|d|}}\Bigg[\frac{S_{r,1}}{|A_{r,1}|^{2g-2+n}}\left(\alpha_0^{(r,1)}+\frac{|A_{r,1}|}{2g-3+n)}\alpha_1^{(r,1)}+\cdots\right)$$

$$+\cdots$$

$$+\frac{S_{r,\lfloor\frac{r-1}{2}\rfloor}}{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^{2g-2+n}}\left(\alpha_0^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\frac{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^K}{2g-3+n}\alpha_1^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\cdots\right)$$

$$+\frac{\delta_r^{\text{even}}}{2}\frac{S_{r,\frac{r}{2}}}{|A_{r,\frac{r}{2}}|^{2g-2+n}}\left(\alpha_0^{(r,\frac{r}{2})}+\frac{|A_{r,\frac{r}{2}}|^K}{2g-3+n}\alpha_1^{(r,\frac{r}{2})}+\cdots\right)\Bigg]$$

where $S_{r,\alpha}$, $A_{r,\alpha}$, $\alpha_k^{(r,\alpha)}$ are obtained the r-Airy ODE.

Thank you for the attention!