String–Math

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Resurgent large genus asymptotics of intersection numbers

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A case study: $m!$

Enumerative problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \mathrm{O}(m^{-1})\right)$

Con: asymptotically exact
Pro: closed-form

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Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + \frac{1}{12} m^{-1} + \frac{1}{288} m^{-2} + O(m^{-3})\right)$

Con: asymptotically exact
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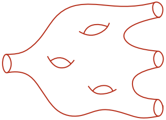
ψ-class intersection numbers

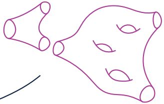
$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i+1)!! \qquad d_1+\cdots+d_n = 3g-3+n$$

- Compute the perturbative expansion of topological 2d gravity
- Feynman diagrams of the Airy matrix model
- Volumes of moduli spaces of metric ribbon graphs
- Building block for all tautological intersection numbers

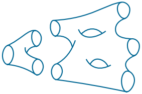
Recursive solution: Virasoro constraints

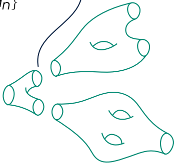
Witten conjecture/Kontsevich theorem, early '90s:





$$\langle\langle \tau_{d_1}\cdots\tau_{d_n}\rangle\rangle = \sum_{m=2}^{n}(2d_m+1)\,\langle\langle \tau_{d_1+d_m-1}\tau_{d_2}\cdots\widehat{\tau_{d_m}}\cdots\tau_{d_n}\rangle\rangle + \frac{1}{2}\sum_{a+b=d_1-2}\left(\langle\langle \tau_a\tau_b\tau_{d_2}\cdots\tau_{d_n}\rangle\rangle + \sum_{\substack{g_1+g_2=g\\ I_1\sqcup I_2=\{d_2,\dots,d_n\}}}\langle\langle \tau_a\tau_{I_1}\rangle\rangle\langle\langle \tau_b\tau_{I_2}\rangle\rangle\right)$$





Virasoro constraints/topological recursion.

Large genus asymptotics

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{\left(\frac{2}{3}\right)^{2g-2+n}} \Big(1 + O(g^{-1})\Big)$$

Proved by Aggarwal (2020), Guo–Yang, (2021)

(combinatorial analysis of Virasoro constraints/determinantal formula)

Questions

- Universal strategy, adaptable to different problems?
- 'Geometric' meaning?
- Subleading corrections?

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Large genus asymptotics: our result

Answers (EGGGL)

- Universal strategy: resurgence + determinantal formula
- Geometric meaning: Airy functions

$$y^2 - x = 0 \qquad \xrightarrow{\textit{quantisation}} \qquad \left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$$

- Subleading corrections: algorithm + properties

Uniformly in $d_1, \ldots, d_n$:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = S \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n}\alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}}\alpha_k + O(g^{-k-1})\right)$$

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$S = 1$
Stokes constant

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$A = 2/3$

$\psi \sim \frac{1}{\sqrt{2}x^{1/4}} e^{\pm \frac{A}{\hbar} x^{-3/2}}$

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Computable; polynomial in n and multiplicities of d_i

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$$\alpha_1 = -\frac{17-15n+3n^2}{12} - \frac{(3-n)(n-p_0)}{2} - \frac{(n-p_0)^2}{4}$$

where $p_0 = \#\{ d_i = 0 \}$

Darboux method

- $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m \xrightarrow{\text{Borel}} \widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$
- Suppose $\widehat{\varphi}$ has log singularities $A_1, \ldots, A_n$:

$$\widehat{\varphi}(s) \sim -\frac{S_i}{2\pi} \widehat{\psi}_i(s - A) \log(s - A_i)$$

S_i are the Stokes constants, $\widehat{\psi}_i(s) = \sum_m \frac{b_{i,m}}{m!} s^m$ are holomorphic

- Large m asymptotics:

$$a_m = \frac{S_1}{2\pi} \frac{\Gamma(m)}{A_1^m} \left(b_{1,0} + \frac{A_1}{m-1} b_{1,1} + \frac{A_1^2}{(m-1)(m-2)} b_{1,2} + \cdots \right)$$
$$+ \cdots$$
$$+ \frac{S_n}{2\pi} \frac{\Gamma(m)}{A_n^m} \left(b_{n,0} + \frac{A_n}{m-1} b_{n,1} + \frac{A_n^2}{(m-1)(m-2)} b_{n,2} + \cdots \right)$$

Darboux method: summary

Upshot:

Borel plane singularities $\Longrightarrow$ large order asymptotics

- Fact 1: Borel plane sings are well-understood for exponential integrals
- Fact 2: Borel plane sings behave well under sums/products

Example: $\mathrm{Ai}(x,\hbar)\cdot\mathrm{Bi}(x,\hbar)$

(the expansion coeff's of Ai and Bi are explicit, but the ones of $\mathrm{Ai}\cdot\mathrm{Bi}$ are not)

Determinantal formula

Take the generating series

$$W_n(x_1,\ldots,x_n;\hbar) = \sum_{g\geqslant 0} \hbar^{2g-2+n} \sum_{d_1,\ldots,d_n} \# \frac{\langle\langle \tau_{d_1}\cdots\tau_{d_n}\rangle\rangle}{x_1^{d_1}\cdots x_n^{d_n}}$$

Det. formula (Bergère–Eynard, Bertola–Dubrovin–Yang):

$$W_n(x_1,\ldots,x_n;\hbar) = \begin{array}{c}\text{sum over permutations of } S_n\\ \text{involving } \mathrm{Ai} \text{ and } \mathrm{Bi}\end{array}$$

Example: $n = 2$

$$W_2 = \frac{\mathrm{Ai}_1\mathrm{Bi}_1\mathrm{Ai}_2'\mathrm{Bi}_2' + \frac{1}{2}\mathrm{Ai}_1\mathrm{Bi}_1'\mathrm{Ai}_2\mathrm{Bi}_2' + \frac{1}{2}\mathrm{Ai}_1\mathrm{Bi}_1'\mathrm{Bi}_2\mathrm{Ai}_2'}{(x_1-x_2)^2} + (x_1 \leftrightarrow x_2)$$

where $\mathrm{Ai}_i = \mathrm{Ai}(x_i,\hbar)$, $\mathrm{Bi}_i = \mathrm{Bi}(x_i,\hbar)$.

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where $\mathrm{Ai}_i = \mathrm{Ai}(x_i,\hbar)$, $\mathrm{Bi}_i = \mathrm{Bi}(x_i,\hbar)$.

Singularity structure of $\widehat{W}_n$

Singularity strct of $\widehat{\mathrm{Ai}}$, $\widehat{\mathrm{Bi}}$ $\Longrightarrow$ Singularity strct of $\widehat{W}_n$

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\tfrac{4}{3}x_i^{3/2} \qquad \text{and} \qquad -\tfrac{4}{3}x_i^{3/2}, \qquad\qquad i = 1, \dots, n$$

- Stokes constants: $S = 1$
- Holom. funct multiplying the $\log$:
  - (A) at $+\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\mathrm{Ai}}_i$ with $\widehat{\mathrm{Bi}}_i$
  - (B) at $-\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\mathrm{Bi}}_i$ with $\widehat{\mathrm{Ai}}_i$

Bessel

Norbury's intersection numbers (super WP/JT, BGW tau function):

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle^{\Theta} = \int_{\overline{\mathcal{M}}_{g,n}} \Theta_{g,n} \prod_{i=1}^{n} \psi_i^{d_i} (2d_i+1)!!$$

$$= S\, \frac{2^n}{4\pi} \frac{\Gamma(2g-2+n)}{A^{2g-2+n}} \left(1 + \frac{A}{2g-3+n}\alpha_1 + \cdots + \frac{A^k}{(2g-3+n)^{\underline{k}}}\alpha_k + O(g^{-k-1}) \right)$$

where:

- $S = 2$

  Stokes constants of the Bessel ODE
- $A = 2$

  leading exp behaviour of K_0
- α_k polynomials in n and multiplicities of d_i

  are computable from the asymptotic expansion coeffs of K_0

r-Airy

Witten's r-spin intersection numbers (FJRW theory, top. gravity coupled to a WZW theory):

$$\langle\langle \tau_{a_1,d_1}\cdots\tau_{a_n,d_n}\rangle\rangle^{r\text{-spin}} = \int_{\overline{\mathcal{M}}_{g,n}} c_W(a_1,\ldots,a_n)\prod_{i=1}^{n}\psi_i^{d_i}(rd_i+a_i)!_{(r)}$$

$$= \frac{2^n}{2\pi}\frac{\Gamma(2g-2+n)}{r^{g-1-|d|}}\Bigg[\frac{S_{r,1}}{|A_{r,1}|^{2g-2+n}}\left(\alpha_0^{(r,1)}+\frac{|A_{r,1}|}{2g-3+n}\alpha_1^{(r,1)}+\cdots\right)$$

$$+\cdots$$

$$+\frac{S_{r,\lfloor\frac{r-1}{2}\rfloor}}{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^{2g-2+n}}\left(\alpha_0^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\frac{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^K}{2g-3+n}\alpha_1^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\cdots\right)$$

$$+\frac{\delta_r^{\text{even}}}{2}\frac{S_{r,\frac{r}{2}}}{|A_{r,\frac{r}{2}}|^{2g-2+n}}\left(\alpha_0^{(r,\frac{r}{2})}+\frac{|A_{r,\frac{r}{2}}|^K}{2g-3+n}\alpha_1^{(r,\frac{r}{2})}+\cdots\right)\Bigg]$$

where $S_{r,\alpha}$, $A_{r,\alpha}$, $\alpha_k^{(r,\alpha)}$ are obtained from the r-Airy ODE.

Thank you for the attention!

Weil–Petersson volumes?

Weil–Petersson volumes satisfy the determinantal formula.

Problem

Understand the WP quantum curve:

$$y^2 - \frac{\sin^2(2\pi\sqrt{x})}{4\pi^2} = 0 \qquad \xrightarrow{\textit{quantisation}} \qquad ??$$

(aka wave/Baker–Akhiezer function)

Visualising the large genus asymptotics

$$\frac{2g-3+n}{2/3}\left(\frac{\langle\langle \tau_{d_1}\cdots\tau_{d_n}\rangle\rangle}{\frac{2^n}{4\pi}\frac{\Gamma(2g-2+n)}{(2/3)^{2g-2+n}}}-1\right)=\alpha_1(n,p_0)+O(g^{-1})$$

For $n=2$: