University of Science and Technology of China

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Resurgent large genus asymptotics of intersection numbers

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A case study: $m!$

Counting problem: $c_m = \#\left\{ \begin{array}{c} \text{arrangements of } m \text{ distinct objects} \\ \text{into } m \text{ distinct boxes} \end{array} \right\}$

Solution: $c_m = m! = \begin{cases} m \cdot c_{m-1} & m > 1 \\ 1 & m = 1 \end{cases}$

Pro: exact
Con: recursive

Asymptotics: $c_m = \sqrt{2\pi m}\left(\frac{m}{e}\right)^m \left(1 + O(m^{-1})\right)$

Con: asymptotically exact
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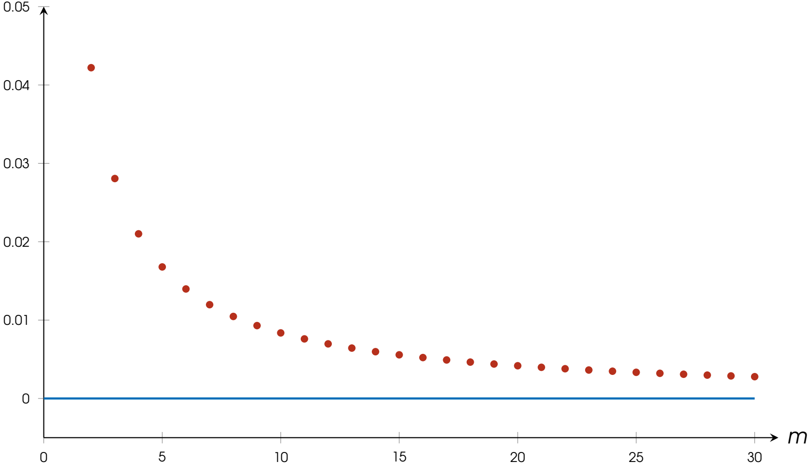
Visualising Stirling's formula

$$\frac{m!}{\sqrt{2\pi m}\left(\frac{m}{e}\right)^m} - 1 = O(m^{-1})$$



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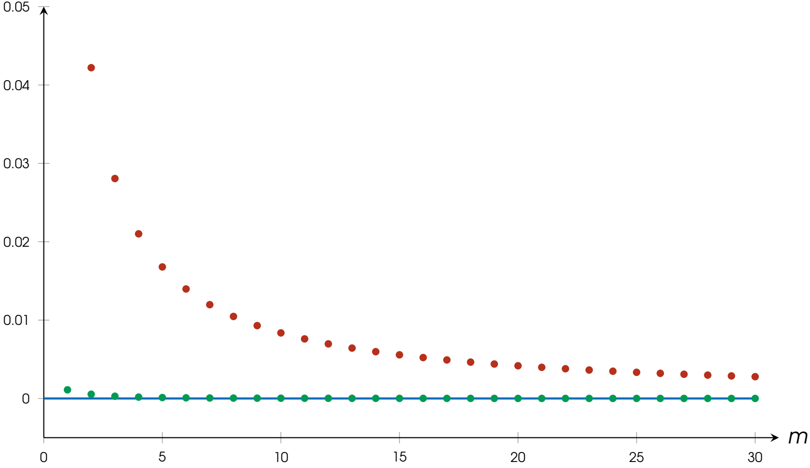

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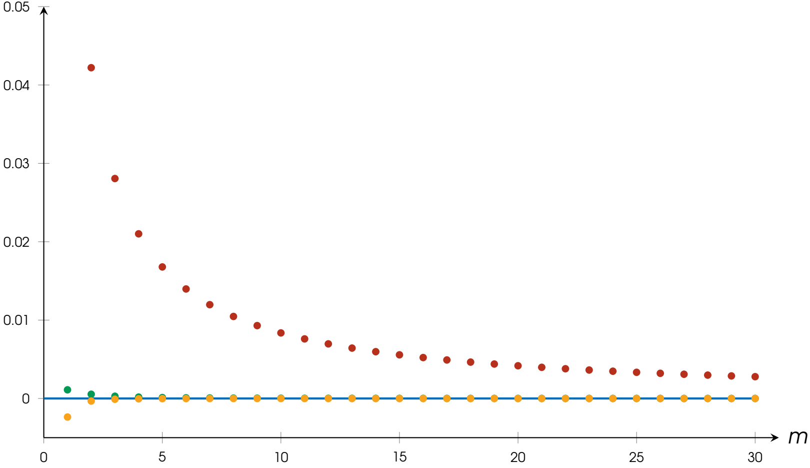

ψ-class intersection numbers

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle = \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{d_1} \cdots \psi_n^{d_n} \qquad d_i \geqslant 0, \qquad d_1 + \cdots + d_n = 3g-3+n$$

- Volumes of moduli spaces of metric ribbon graphs
- Building block for all tautological intersection numbers:
  - Weil–Petersson volumes
  - Masur–Veech volumes
  - Hurwitz numbers
  - Gromov–Witten invariants for targets with s.s. quantum cohomology
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- Compute the perturbative expansion of topological 2d gravity

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Solution

Normalisation: $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} (2d_i + 1)!!$

Witten conjecture/Kontsevich theorem, early '90s:

$$\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle = \sum_{m=2}^{n} (2d_m + 1) \langle\langle \tau_{d_1+d_m-1} \tau_{d_2} \cdots \widehat{\tau_{d_m}} \cdots \tau_{d_n} \rangle\rangle + \frac{1}{2} \sum_{a+b=d_1-2} \left(\langle\langle \tau_a \tau_b \tau_{d_2} \cdots \tau_{d_n} \rangle\rangle + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{d_2,\ldots,d_n\}}} \langle\langle \tau_a \tau_{I_1} \rangle\rangle \langle\langle \tau_b \tau_{I_2} \rangle\rangle \right)$$

with initial data $\langle\langle \tau_0^3 \rangle\rangle = 1$ and $\langle\langle \tau_1 \rangle\rangle = \frac{1}{8}$.

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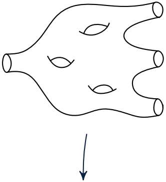

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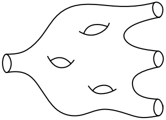

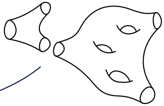

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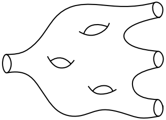

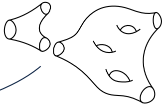

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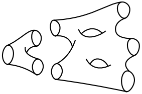

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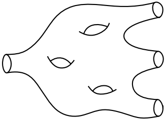

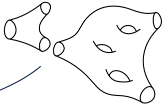

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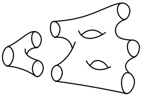

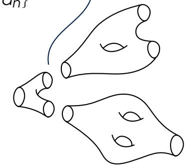

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Large genus asymptotics

Uniformly in $d_1, \dots, d_n$ as $g \to \infty$:

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- Proved by Aggarwal, 2020
  (combinatorial analysis of Witten–Kontsevich topological recursion)
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Questions

- Universal strategy, adaptable to different problems?
- 'Geometric' meaning of the formula?
- Subleading corrections?

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Large genus asymptotics: new perspective

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- Geometric meaning: Airy functions

$$y^2 - x = 0 \xrightarrow{\text{quantisation}} \left(\hbar^2 \frac{d^2}{dx^2} - x\right) \psi(x,\hbar) = 0$$

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$S = 1$
Stokes constant

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$A = 2/3$

$\psi \sim \frac{1}{\sqrt{2}x^{1/4}} e^{\pm \frac{A}{\hbar} x^{-3/2}}$

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Computable; polynomial in n and multiplicities of d_i

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$\alpha_1 = -\frac{17-15n+3n^2}{12} - \frac{(3-n)(n-p_0)}{2} - \frac{(n-p_0)_2}{4}$

where $p_0 = \#\{d_i = 0\}$

Visualising the large genus asymptotics

$$\frac{2g-3+n}{2/3}\left(\frac{\langle\langle\tau_{d_1}\cdots\tau_{d_n}\rangle\rangle}{\frac{2^n}{4\pi}\frac{\Gamma(2g-2+n)}{(2/3)^{2g-2+n}}}-1\right)=\alpha_1(n,p_0)+O(g^{-1})$$

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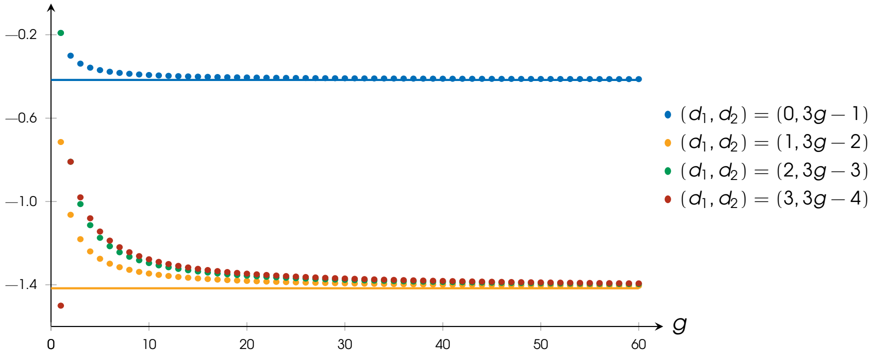

Darboux meets Borel

Darboux's idea:

- Abs. convergent power series:

$$\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$$

- Get a holomorphic function around the origin, take analytic continuation
- The large m asymptotics of a_m is totally controlled by the behaviour of $\widehat{\varphi}$ at its singularities

Borel's idea:

- Divergent power series:

$$\widetilde{\varphi}(\hbar) = \sum_{m \geqslant 0} a_m \hbar^m, \quad |a_m| = O(A^{-m} m!)$$

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Darboux's result: sketch of the proof

Take a convergent power series: $\widehat{\varphi}(s) = \sum_{m \geqslant 0} \frac{a_m}{m!} s^m$

Suppose its analytic continuation has a single log singularity at $s = A$:

$$\widehat{\varphi}(s) = (\text{holomorphic } @A) \log(s - A) + \text{holomorphic } @A$$

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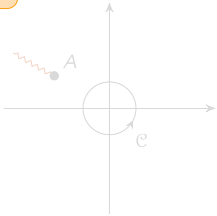

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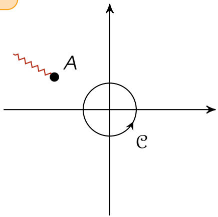

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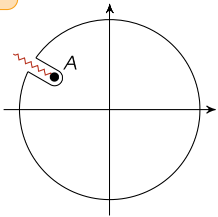

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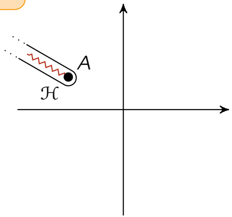

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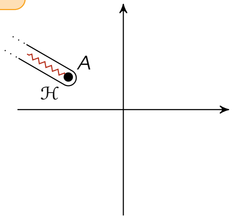

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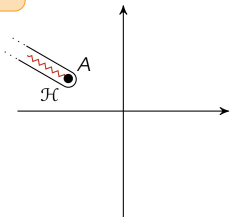

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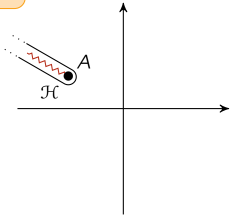

Darboux meets Borel: summary

- Given: $\widetilde{\varphi}(\hbar) = \sum_m a_m \hbar^m$ divergent
- Borel transform: $\widehat{\varphi}(s) = \sum_m \frac{a_m}{m!} s^m$ convergent
- Suppose you can compute:
  1. Log singularities: $A_1, \ldots, A_n$
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  3. Minors: $\widehat{\psi}_1, \ldots, \widehat{\psi}_n$
- Large m asymptotics:

$$a_m = \frac{\Gamma(m)}{2\pi}\left(\frac{S_1}{A_1^m}\left(b_{1,0} + \frac{A_1}{m-1}b_{1,1} + \frac{A_1^2}{(m-1)(m-2)}b_{1,2} + \cdots\right) + \cdots + \frac{S_n}{A_n^m}\left(b_{n,0} + \frac{A_n}{m-1}b_{n,1} + \frac{A_n^2}{(m-1)(m-2)}b_{n,2} + \cdots\right)\right)$$

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Main example: Airy function

The formal (WKB) solutions of the Airy ODE, $\left(\hbar^2 \frac{d^2}{dx^2} - x\right)\psi(x,\hbar) = 0$, are

$$\psi_{\mathrm{Ai}}(x;\hbar) = \frac{e^{-\frac{2}{3\hbar}x^{3/2}}}{\sqrt{2}x^{1/4}}\left(1 - \frac{5}{48x^{3/2}}\hbar + \frac{385}{4608x^3}\hbar^2 + \cdots\right)$$

$$\psi_{\mathrm{Bi}}(x;\hbar) = \psi_{\mathrm{Ai}}(x;-\hbar)$$

$\widetilde{\psi}_{\mathrm{Ai}}$ is a divergent series in $\hbar$. Its Borel transform has a single log singularity at $s = +\frac{4}{3}x^{3/2}$:

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Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} W_{g,n}(x_1, \ldots, x_n)$$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure

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$\hbar \to$ genus
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$\hbar \to$ genus
$x_i \to d_i$

2. W_n is a divergent series in $\hbar$. Take its Borel transform and study its singularity structure

3. Get the large genus asymptotics (with subleading contributions!)

Strategy towards large genus asymptotics

Goal

Compute the large genus asymptotics of $\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle$

1. Take the generating series

$$W_n(x_1, \ldots, x_n; \hbar) = \sum_{g \geqslant 0} \hbar^{2g-2+n} W_{g,n}(x_1, \ldots, x_n)$$

$$= (-2)^{-(2g-2+n)} \sum_{d_1, \ldots, d_n} \frac{\langle\langle \tau_{d_1} \cdots \tau_{d_n} \rangle\rangle}{2x_1^{d_1+3/2} \cdots 2x_n^{d_n+3/2}}$$

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3. Get the large genus asymptotics (with subleading contributions!)

Determinantal formula: setup

Arrange the formal Airy functions as

$$\Psi(x,\hbar) = \begin{pmatrix} \psi_{\mathrm{Ai}} & \psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}} & \psi'_{\mathrm{Bi}} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \,.$$

It solves the system $\left(\hbar\frac{d}{dx} - \left(\begin{smallmatrix} 0 & 1 \\ x & 0 \end{smallmatrix}\right)\right)\Psi = 0$.

Define the matrix $M = \Psi \frac{1}{2} \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \Psi^{-1}$:

$$M(x,\hbar) = \begin{pmatrix} \frac{1}{2}(\psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}} + \psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}}) & \psi_{\mathrm{Ai}}\psi_{\mathrm{Bi}} \\ \psi'_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} & -\frac{1}{2}(\psi_{\mathrm{Ai}}\psi'_{\mathrm{Bi}} + \psi'_{\mathrm{Ai}}\psi_{\mathrm{Bi}}) \end{pmatrix} \in \mathfrak{sl}(2,\mathbb{C}) \,.$$

Crucial facts:

- M contains only quadratic products Ai-Bi
- The exponential terms cancel out $\implies$ M is a formal series in $\hbar$.

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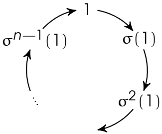
Cyclic permutations:



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Example: $n = 2$

$$W_2 = -\frac{\operatorname{tr}\big(M(x_1,\hbar)M(x_2,\hbar)\big)}{(x_1 - x_2)(x_2 - x_1)}$$

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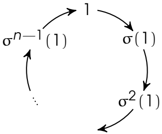

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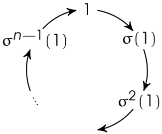

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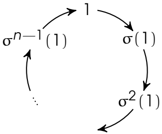

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Singularity structure of $\widehat{W}_n$

Singularity strct of $\widehat{\psi}_{\mathrm{Ai}}, \widehat{\psi}_{\mathrm{Bi}}$ $\Longrightarrow$ Singularity strct of $\widehat{W}_n$

- $2n$ log singularities of $\widehat{W}_n$, located at

$$+\tfrac{4}{3}x_i^{3/2} \quad \text{and} \quad -\tfrac{4}{3}x_i^{3/2}, \qquad i = 1, \ldots, n$$

- Stokes constants: $S = 1$
- Minors:
  - (A) at $+\frac{4}{3}x_i^{3/2}$: replace each $\widehat{\psi}_{\mathrm{Ai}}(x_i; s)$ with $\widehat{\psi}_{\mathrm{Bi}}(x_i; s)$
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Summary

Uniformly in $d_1, \ldots, d_n$ as $g \to \infty$:

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where:

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$$+\frac{S_{r,\lfloor\frac{r-1}{2}\rfloor}}{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^{2g-2+n}}\left(\alpha_0^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\frac{|A_{r,\lfloor\frac{r-1}{2}\rfloor}|^K}{2g-3+n}\alpha_1^{(r,\lfloor\frac{r-1}{2}\rfloor)}+\cdots\right)$$

$$+\frac{\delta_r^{\text{even}}}{2}\frac{S_{r,\frac{r}{2}}}{|A_{r,\frac{r}{2}}|^{2g-2+n}}\left(\alpha_0^{(r,\frac{r}{2})}+\frac{|A_{r,\frac{r}{2}}|^K}{2g-3+n}\alpha_1^{(r,\frac{r}{2})}+\cdots\right)\Bigg]$$

where $S_{r,i}$, $A_{r,i}$, $\alpha_k^{(r,i)}$ are obtained the r-Airy ODE.

谢谢你的注意力!